

Bryn Mawr College Scholarship, Research, and Creative Work at Bryn Mawr College

Bryn Mawr College Dissertations and Theses

2008

Legendrian Torus Links

Jennifer Dalton

Bryn Mawr College, j.dalton@rivers.org

Follow this and additional works at: <https://repository.brynmawr.edu/dissertations>



Part of the [Mathematics Commons](#)

Custom Citation

Dalton, Jennifer. "Legendrian Torus Links." PhD diss., Bryn Mawr College, 2008.

This paper is posted at Scholarship, Research, and Creative Work at Bryn Mawr College. <https://repository.brynmawr.edu/dissertations/6>

For more information, please contact repository@brynmawr.edu.

Contents

1	Introduction	2
2	Background	18
2.1	Basic contact geometry and Legendrian knots	18
2.2	Convex Surfaces	23
2.3	Algebraic Invariants	38
3	Positive Torus Links	55
4	Negative Torus Links	71
5	A <i>DGA</i> Approach to Noncyclic Permutations	87
5.1	General Strategy	87
5.2	Set up and Labeling of Diagrams	90
5.3	Permutations of Negative $(p, 2)$ -Torus Links	96
5.4	Permutations of the $(-4, 3)$ -Torus Link	100
5.5	Other Negative (p, q) -Torus Links	123

Chapter 1

Introduction

A basic problem in contact topology is to determine whether two Legendrian knots or links are equivalent. In this dissertation, we will study the equivalence and non-equivalence of Legendrian links that are topologically torus links.

First recall the basic construction of torus knots and links. A standardly embedded torus T provides a genus one Heegard splitting of S^3 , $S^3 = V_0 \cup_T V_1$ where V_0 and V_1 are solid tori. Then any curve on T can be written as $p\mu + q\lambda$ where μ is the unique curve that bounds a disk in V_0 , and λ is the unique curve that bounds a disk in V_1 . We orient μ arbitrarily and then orient λ so that μ, λ form a positive basis for $H_1(T)$ where T is oriented as the boundary of V_0 . Up to homotopy, any curve on T can be written as $p\mu + q\lambda$. When p and q are relatively prime, this is a (p, q) -torus knot. Since (p, q) -torus knots agree with $(-p, -q)$ - and (q, p) -torus knots, we will use the convention that

$|p| > q > 0$. Figure 1.1 shows a $(3, 2)$ -torus knot as it sits on a torus. Here we can see that the knot wraps 3 times in the μ direction and 2 times in the λ direction. If $p > 0$ then we have a *positive* torus knot, and if $p < 0$ we have a *negative* torus knot. See Figure 1.2 for examples of a $(3, 2)$ - and a $(-3, 2)$ -torus knot. A torus link comes from looking at (np, nq) where p and q are relatively prime and $n \neq 1$. For such n , p , and q , an (np, nq) -torus link has n components each of which is a (p, q) -torus knot. For a 2-dimensional torus T , we will choose a coordinate system, i.e., an identification of T with $\mathbb{R}^2/\mathbb{Z}^2$, so that the meridian has slope 0, and a specified longitude has slope ∞ . In this way, a (p, q) -torus knot corresponds to a line of slope $\frac{q}{p}$ in $\mathbb{R}^2/\mathbb{Z}^2$, and an (np, nq) -torus link corresponds to n parallel lines of slope $\frac{q}{p}$ in $\mathbb{R}^2/\mathbb{Z}^2$. See Figure 1.1.

Figure 1.1: A $(3, 2)$ -torus knot.

Our focus will be Legendrian versions of torus knots and links. Legendrian knots are knots satisfying additional geometric conditions imposed by a contact structure. We will use the standard contact structure on \mathbb{R}^3 ,

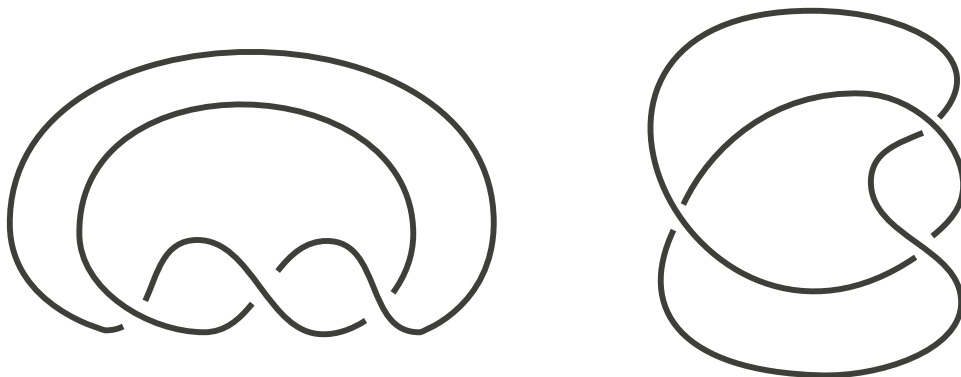


Figure 1.2: *Left:* A $(3, 2)$ -torus knot. *Right:* A $(-3, 2)$ -torus knot.

$\xi = \ker(dz - ydx)$ (see Figure 1.3). A *Legendrian curve* is a curve L where $T_p L \subset \xi_p, \forall p \in L$. If L is closed then it is a *Legendrian knot*. A *Legendrian link* is a disjoint union of Legendrian knots.

We will represent Legendrian knots and links in (\mathbb{R}^3, ξ) via the *front projection*. Let

$$\begin{aligned} \Pi : \mathbb{R}^3 &\rightarrow \mathbb{R}^2 \\ (x, y, z) &\mapsto (x, z). \end{aligned}$$

The image, $\Pi(L)$, of a Legendrian curve L is called the *front projection* of L . Figure 1.4 gives the front diagrams for Legendrian versions of the $(3, 2)$ – and $(-3, 2)$ -torus knots of Figure 1.2. Note that the front diagram of a Legendrian knot will never have vertical tangencies or self tangencies. In addition, at each crossing the arc with the lesser slope will lie over the other arc; this is because the third coordinate is given by $y = \frac{dz}{dx}$, and to keep a right-handed coordinate system the positive y -axis points into the page.

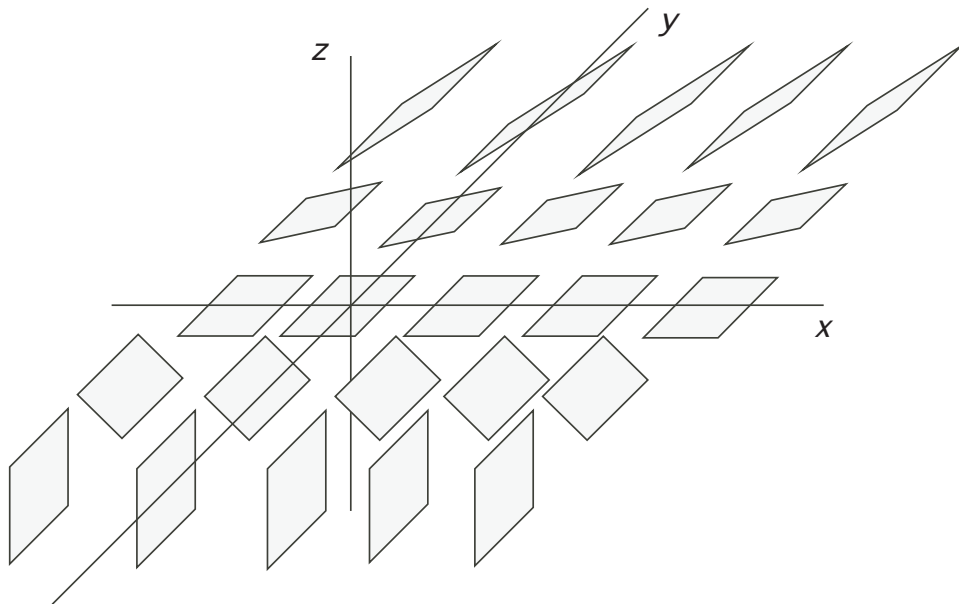
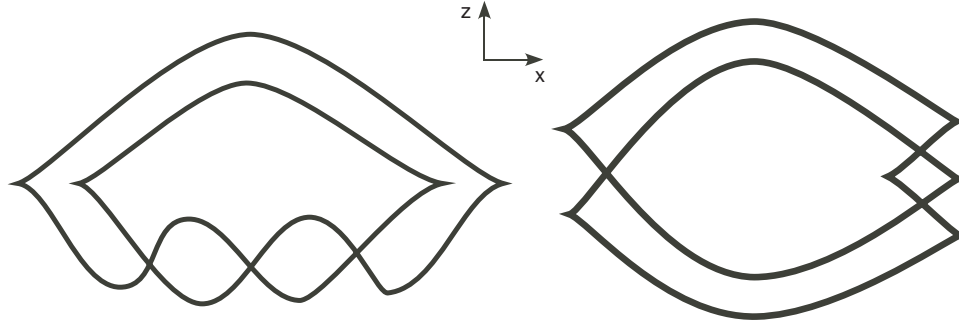


Figure 1.3: The standard contact structure ξ . As $y \rightarrow \pm\infty$, the planes approach but never equal the “vertical” yz -plane.

Every knot and link type, specifically torus knots and links, will have a Legendrian realization. We are interested in whether or not two Legendrian versions of a fixed topological type can be connected by a 1-parameter family of Legendrian knots. There are two classical invariants that can be easily computed from the front projection. These are the Thurston-Bennequin and the rotation number invariants. The combinatorial formula for the Thurston-Bennequin invariant of an oriented Legendrian knot is

$$tb(L) = P - N - \frac{C}{2},$$

where P is the number of positive crossings, N is the number of negative

Figure 1.4: Legendrian versions of the $(3, 2)$ -Torus knots in Figure 1.2.

crossings, and C is the number of cusps. Note that $\frac{C}{2}$ is the same as the number of right cusps. (See Figure 1.5.) The combinatorial formula for the rotation number of an oriented Legendrian knot is

$$r(L) = \frac{1}{2}(D - U)$$

where D is the number of down cusps, and U is the number of up cusps. (See Figure 1.6.) Alternate definitions of these classical invariants are given in Chapter 2.

$$tb(K) = \# \begin{array}{c} \nearrow \\ \searrow \end{array} + \# \begin{array}{c} \nwarrow \\ \swarrow \end{array} - \# \begin{array}{c} \nearrow \\ \swarrow \end{array} - \# \begin{array}{c} \nwarrow \\ \searrow \end{array} - \# \begin{array}{c} \curvearrowright \end{array}$$

Figure 1.5: The Thurston-Bennequin invariant can easily be computed by counting the number of positive and negative crossings and the number of right cusps in a front projection of the knot.

From these invariants, it is easy to see that any topological knot type has an infinite number of nonequivalent Legendrian realizations through “sta-

$$r(K) = \frac{1}{2} (\# \begin{array}{c} \nearrow \\ \searrow \end{array} + \# \begin{array}{c} \nwarrow \\ \nearrow \end{array} - \# \begin{array}{c} \nearrow \\ \nearrow \end{array} - \# \begin{array}{c} \nwarrow \\ \nwarrow \end{array})$$

Figure 1.6: The rotation number can easily be computed by counting the number of down and up cusps in the front projection of the knot.

bilization.” A *stabilization* of a Legendrian knot K is obtained by adding a “zig-zag” to the front projection of the knot and is denoted $S_{\pm}(K)$ (see Figure 1.7). The \pm is determined by whether the rotation number of the stabilized knot is greater or less than that of K . We also get that $tb(S_{\pm}(K)) = tb(K) - 1$. Since K and $S_{\pm}(K)$ have different rotation numbers and Thurston-Bennequin invariants, they are not Legendrian isotopic. However, they are in the same knot type. In this way, we see that there are an infinite number of Legendrian realizations of a knot in a fixed knot type. In Chapter 2, a description of stabilization and destabilization is described in terms of bypass disks.

Given a fixed knot type, there is a maximum for the Thurston-Bennequin invariant that can be realized by Legendrian knots in this knot type. This maximal Thurston-Bennequin invariant for torus knots was studied in [5].

There are a few knots whose Legendrian classifications are completely understood in terms of the classical Thurston-Bennequin and rotation number invariants. The following result says that all Legendrian unknots descend from a unique one with maximal Thurston-Bennequin invariant. This relationship can be seen in Figure 1.7. In this figure, the knots are oriented clockwise, positive stabilizations are to the right, and negative stabilizations are to the left.

Theorem 1.0.1. (Eliashberg and Fraser, [4]) *In any tight contact three manifold, Legendrian unknots are determined by their Thurston-Bennequin invariant and rotation number. All Legendrian unknots are stabilizations of the unique one with $tb = -1$ and $r = 0$.*

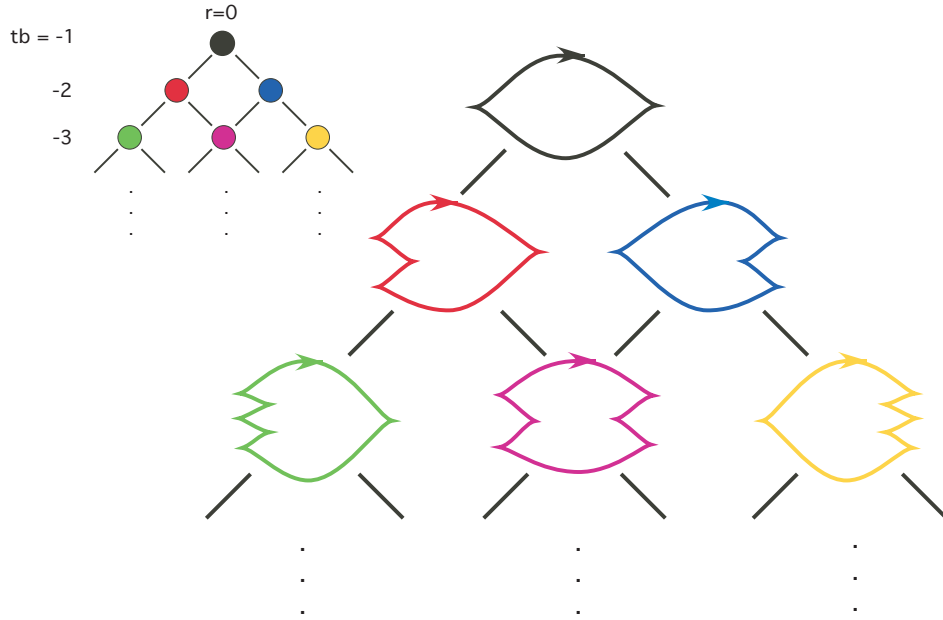


Figure 1.7: The tree of Legendrian unknots.

An unknot is a simple torus knot. In 2001, Etnyre and Honda found a complete classification of all Legendrian torus knots.

Theorem 1.0.2. (Etnyre and Honda, [9]) *In any tight contact three manifold, Legendrian torus knots are determined up to Legendrian isotopy by their knot type, Thurston-Bennequin invariant, and rotation number. Moreover, the precise range of the classical invariants is given as follows:*

1. All Legendrian positive (p, q) -torus knots are stabilizations of the unique one with $tb = pq - p - q$ and $r = 0$.
2. All Legendrian negative (p, q) -torus knots are stabilizations of one with maximal $tb = pq$. Moreover, if $|p| = mq + e$, $0 < e < q$, there are $2m$ torus knots with maximal Thurston-Bennequin invariant, and the set of realized rotation numbers is

$$\{\pm(p + q + 2nq) | 0 \leq n \leq m - 1\}.$$

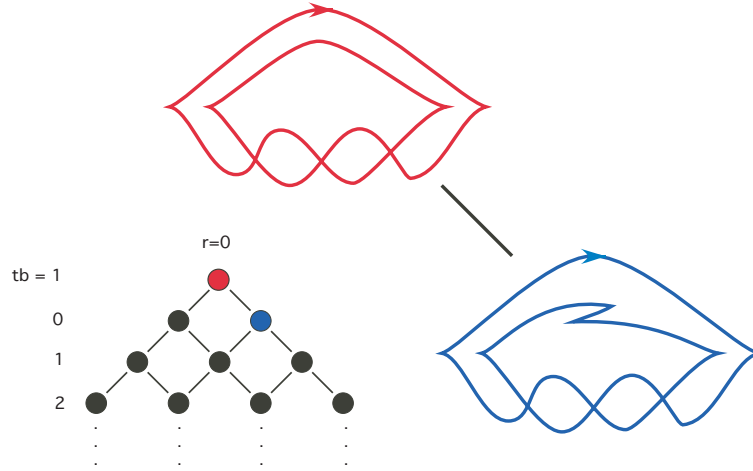


Figure 1.8: All $(3, 2)$ Torus knots are stabilizations of the unique one with maximal Thurston-Bennequin invariant.

As a consequence, all positive torus links can be represented with a single peaked tree diagram like the one for unknots, with the peak at $r = 0$, $tb = pq - p - q$; Figure 1.8 represents all $(3, 2)$ -torus knots. For negative

torus knots, there are multiple knots with maximal Thurston-Bennequin invariant. For negative torus links, the relationship described in the above theorem can be seen by a “mountain range” diagram with $2m$ peaks like the one in Figure 1.9 for $(-5, 2)$ -torus knots. Notice that each knot with maximal Thurston-Bennequin invariant has a set of stabilizations that may overlap with stabilizations of other knots with maximal Thurston-Bennequin invariant. By this theorem, each vertex in Figures 1.8 and 1.9 represents only one knot. In particular, different stabilizations of different knots may give equivalent knots.

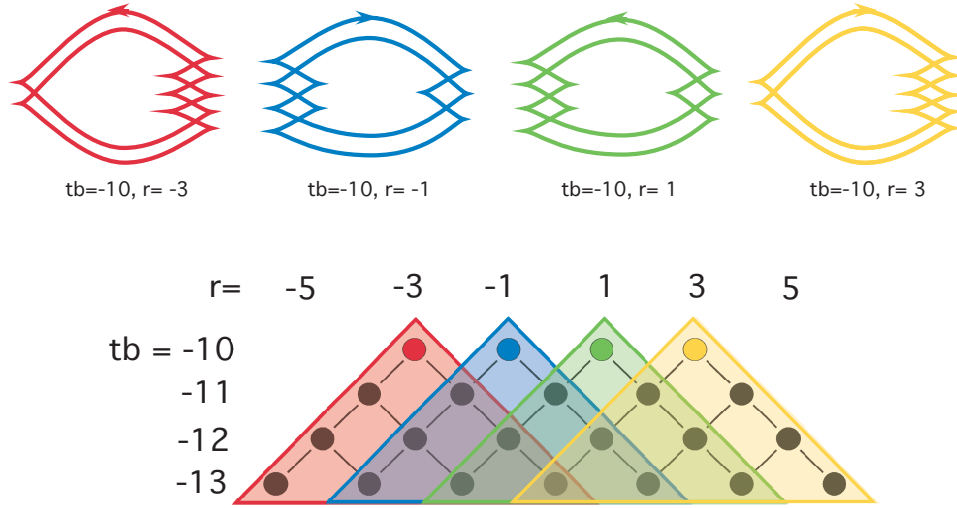


Figure 1.9: All possible $(-5, 2)$ torus knots are stabilizations of one with maximal Thurston-Bennequin invariant.

In this dissertation, we study the classification of both ordered and unordered Legendrian torus links with knotted components, that is (np, nq) -torus links where $n \geq 2$, $\gcd(p, q) = 1$, $|p| \geq q > 1$. Recall that such an

(np, nq) -torus link will have n components, each of which is a (p, q) -torus knot. Here are some natural questions that arise when studying Legendrian torus links:

Legendrian Torus Link Questions:

1. *Is it possible to construct a Legendrian (np, nq) -torus link using any n Legendrian (p, q) -torus knots?*
2. *A clearly necessary condition for the equivalence of two Legendrian torus links L and L' that represent the same topological link is that we be able to pair up each component of L with one from L' so that the elements of each pair have the same Thurston-Bennequin and rotation number invariants. Is it possible to have two different Legendrian (np, nq) -torus links having components with pairwise identical Thurston-Bennequin and rotation number invariants?*
3. *Are the unordered and ordered classifications of Legendrian torus links different?*

We will address these questions for all positive torus links and for negative torus links of knotted components. The following theorem shows that the answer to (1) is yes for Legendrian positive torus links and no for Legendrian negative torus links; we also see that for *unordered* Legendrian torus links, the answer to (2) is no for both positive and negative torus links.

Theorem 1.0.3. (Unordered Torus Link Classification) *Let L be an unordered Legendrian (np, nq) -torus link with components K_1, \dots, K_n , and L'*

be an unordered Legendrian (np, nq) -torus link with components K'_1, \dots, K'_n . Then L and L' are Legendrian isotopic if and only if there exists a permutation σ of $\{1, 2, \dots, n\}$ such that $tb(K_i) = tb(K'_{\sigma(i)})$ and $r(K_i) = r(K'_{\sigma(i)})$. Moreover, the precise range of the classical invariants is given as follows:

1. For positive torus links ($p > 0$), there exists a unique positive (np, nq) -torus link with each component having maximal Thurston-Bennequin invariant; that is, all components will have $tb = pq - p - q$. Any positive torus link with any component having non-maximal Thurston-Bennequin invariant will destabilize to the one with maximal Thurston-Bennequin invariant in all components.
2. For negative torus links ($p < 0, q \neq 1$), if $|p| = mq + e$, $0 < e < q$, there are $2m$ Legendrian realizations of the (np, nq) -torus link with maximal Thurston-Bennequin invariant in all components; that is, all components will have $tb = pq$ and the same rotation number. Any negative torus link with non-maximal Thurston-Bennequin invariant in any component will destabilize to one with maximal Thurston-Bennequin invariant in all components.

Since an (np, nq) -torus link has n components each of which is a (p, q) -torus knot, a useful way to visualize an (np, nq) -torus link is as a collection of n vertices on the tree that represents the possible (p, q) -torus knots. For example, there is a one-to-one correspondence between $(6, 4)$ -torus links and pairs of vertices on the $(3, 2)$ -tree. This means that any two vertices on the

$(3, 2)$ tree in Figure 1.8 can be used to construct a $(6, 4)$ -torus link, and also that any $(6, 4)$ -torus link can be represented as two vertices on this tree. However, for negative torus links, the above theorem indicates that there are more restrictions on the collection of n vertices that correspond to torus links. For example, in Figure 1.9, only vertices in the same colored triangle can be linked together to form a $(-5n, 2n)$ -torus link. In particular, the red and green knots shown with maximal Thurston-Bennequin invariant and rotation numbers -3 and -1 respectively cannot form a $(-10, 4)$ -torus link, but they can each be combined with the vertex representing the knot with $r = -1$ and $tb = -12$ to make a $(-10, 4)$ -torus link.

Remark 1.0.4. Legendrian negative torus links with unknotted components ($q = 1$) follow a different pattern than Theorem 1.0.3. For negative (np, n) -torus links, it is possible to have a link that maximizes the sum of the Thurston-Bennequin invariants of the components where all components do not have the same Thurston-Bennequin invariant. For example, Figure 1.10 is an example of a $(-4, 2)$ -torus link where the sum of the Thurston-Bennequin invariants of the components is maximized. In addition, there are examples of Legendrian negative torus links with unknotted components that do not have maximal Thurston-Bennequin invariant and do not destabilize. The classification of these links has been studied and appears in [2]. \diamond

Theorem 1.0.3 is proved using convex surface techniques. Background on convex surfaces is given in Section 2.2. The proof of this theorem will be

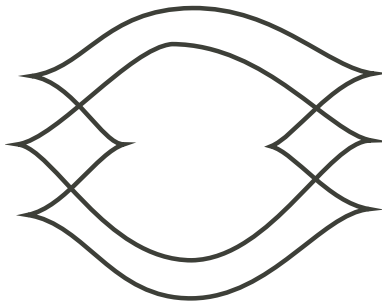


Figure 1.10: A $(-4, 2)$ -torus link with maximal Thurston-Bennequin invariant. Note that the components have different Thurston-Bennequin invariants.

given in two parts since there are significant differences between the proof for positive torus links and for negative torus links. The proof for positives is given in Chapter 3, and the proof for negatives is given in Chapter 4.

We next turn to the question of the ordered classification of Legendrian torus links. In the ordered classification, each component of the link is assigned a “color,” and it is natural to ask if there exists an isotopy of the link starting and ending at the same unordered link, but when colors are assigned to the components the isotopy permutes the colors. Note that it is possible to topologically permute any components of an (np, nq) -torus link. This can be easily seen using a figure like Figure 1.1 by lifting a line in $\mathbb{R}^2/\mathbb{Z}^2$, shifting it slightly, and projecting back to $\mathbb{R}^2/\mathbb{Z}^2$. For Legendrian torus links, we have the following definition.

Definition 1.0.5. Let L and L' be ordered (np, nq) -torus links with components K_1, \dots, K_n and K'_1, \dots, K'_n respectively. Given a permutation σ of $\{1, \dots, n\}$ such that $tb(K_i) = tb(K'_{\sigma(i)})$ and $r(K_i) = r(K'_{\sigma(i)})$, an *invariant*

preserving permutation of the components of L is a contact isotopy φ_t of \mathbb{R}^3 such that $\varphi_0 = id$ and $\varphi_1(K_i) = K'_{\sigma(i)}$ for all i .

We see that there is a lot of flexibility in the ordering of components for positive torus links. For negative torus links, there is some flexibility once the components have been stabilized.

Theorem 1.0.6. *It is possible to do all invariant preserving permutations of the components of a positive (np, nq) -torus link. For negative (np, nq) -torus links, any invariant preserving permutation that preserves the cyclic ordering of the components with $tb = pq$ is possible.*

The above theorem does not address the ordering question when $tb = pq$. We have reason to believe that when $tb = pq$, it is not possible to do noncyclic permutations of the components. This belief is in part due to the results of Mishachev in 2002. Mishachev studied orderings of torus links with unknotted components, (np, n) , for $p < 0$. [17] The “ n -copy” of a Legendrian unknot with $tb = p < 0$ is an example of a (np, n) -torus link. In particular, $(-n, n)$ is n copies of the unknot with maximal $tb = -1$ shifted slightly off of itself in the z direction. Mishachev proved that it is not possible to arbitrarily permute the components of $(-n, n)$. In fact, he showed that cyclic permutations are possible, but non-cyclic permutations are not possible (see Figure 1.11). He then proved the following more general result:

Theorem 1.0.7. (Mishachev, [17]) *Only cyclic permutations are possible for the n -copy of a topologically trivial Legendrian knot.*

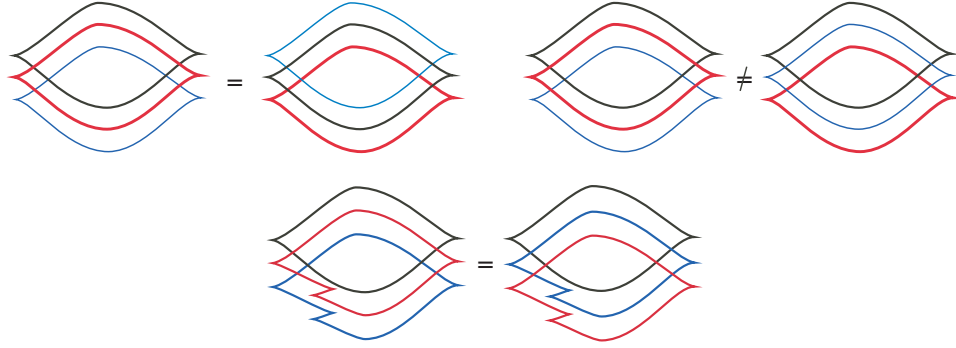


Figure 1.11: The top row shows cyclic (*left*) and non-cyclic permutations (*right*) of $(-3, 3)$. The links on the bottom row illustrate that non-cyclic permutations are possible for stabilized components.

Mishachev proved his result that noncyclic permutations of the n -copy of the unknot with $tb = p$ are not possible by studying invariants that could be extracted from a differential graded algebra (DGA) associated to the link. This invariant depends upon a calculation for each augmentation class of the algebra. In Mishachev's setting, there was a unique augmentation. When $q = 2$, there is again a unique augmentation, but the invariant cannot differentiate the orderings of the components of the link. It is then necessary to study permutations of the double of the knot. The DGA associated to the double has many augmentations, and an unknown number of augmentation classes. When $q > 2$, it is again necessary to study the double of the knot. As with $q=2$, there are many augmentations, and an unknown number of augmentation classes. In Chapter 5, we outline a strategy for distinguishing the links using a calculation for each augmentation, and show how to use these calculations to say enough about the augmentation classes to distin-

guish the links. The strategy is illustrated using $(-4, 3)$ as an example. Parts of this strategy are proved rigorously for $(-4, 3)$. Other parts involve conjectures resulting from calculations that have been done by hand. Since these calculations are long and cumbersome, not all of the details are included in this dissertation, although the results of these calculations are highlighted for each step of the strategy. Nevertheless, the example demonstrates what needs to be shown in order to distinguish the links, and we believe that non-cyclic permutations are not possible for the n -copy of $(-4, 3)$. As of now, we further believe that the result holds for any value of q , but proceeding as with the $q = 3$ case is not practical without the use of a computer. If more was understood about augmentation theory, it might be possible to simplify these calculations. This is a direction for future research.

Chapter 2

Background

2.1 Basic contact geometry and Legendrian knots

A *contact structure* on a 3-manifold M is a maximally nonintegrable plane field ξ . In this dissertation, we are studying links that lie on a 2-dimensional torus. Given such a surface, it is natural to study the foliation induced by the contact structure. In general, if Σ is a surface in M then $\xi \cap T\Sigma$ is a singular line field on Σ and may be integrated to a singular foliation Σ_ξ called the *characteristic foliation*. The characteristic foliation tells us a great deal of information; in particular it determines a contact structure in a neighborhood of the surface. However, the characteristic foliation is very delicate. Moving the surface slightly can dramatically change the foliation. We will see later when discussing convex surfaces that essential information about the contact

structure can be encoded with objects simpler than a vector field.

The links we are studying are all in \mathbb{R}^3 with the standard contact structure $\xi = \ker(dz - ydx)$ (see Figure 1.3). This is an example of a “tight” contact structure. A contact structure ξ is called *tight* if there are no embedded disks D with a limit cycle in their characteristic foliation. If ξ is not tight then it is called *overtwisted*. As a step towards understanding the classification of Legendrian torus knots and links, we must first understand the classification of contact structures on solid tori. As a step towards that, we make use of the following two results concerning the classification of tight contact structures on the 3-ball.

Theorem 2.1.1. (Eliashberg [3].) *A tight contact structure on the 3-ball is uniquely determined (up to isotopy) by the characteristic foliation on its boundary.*

In this same paper, Eliashberg studied the group of contactomorphisms of S^3 with respect to its standard contact structure ξ_0 . Fix a point $p \in S^3$. Let $Diff_0(S^3)$ be the group of orientation-preserving diffeomorphisms of S^3 that fix the plane $\xi_0(p)$. Let $Diff_{\xi_0}$ be the subgroup of $Diff_0(S^3)$ that preserves ξ_0 .

Theorem 2.1.2. (Eliashberg [3].) *The natural inclusion of*

$$Diff_{\xi_0} \hookrightarrow Diff_0(S^3)$$

is a weak homotopy equivalence.

In particular, Theorem 2.1.2 says that the inclusion induces an isomorphism of homotopy groups. Since $Diff_0(S^3)$ is homotopy equivalent to $SO(4)$, we know that $Diff_{\xi_0}$ is path connected. So any orientation preserving contact diffeomorphism that fixes a point is contact isotopic to the identity. In later chapters, we will apply this theorem to show that if there is a contact diffeomorphism taking, for example, a torus T_1 to a torus T_2 , then there is a contact isotopy taking T_1 to T_2 . As an illustration, see the proof of Lemma 2.2.4 later in this section.

A curve γ in M is called a *Legendrian curve* if it is everywhere tangent to the contact plane field ξ . If γ is closed, then it is called a *Legendrian knot*.

The first invariant of a Legendrian knot is its underlying topological knot type. Given a Legendrian knot L we denote its underlying knot type by $k(L)$. Let \mathcal{K} denote a topological knot type, then the set of all Legendrian knots with this knot type is $\mathcal{L}(\mathcal{K}) = \{L | k(L) \in \mathcal{K}\}$.

We define the twisting number $tw(\gamma, \mathcal{F})$ of a closed Legendrian curve γ with respect to a given framing \mathcal{F} to be the number of counterclockwise 2π twists of ξ along γ relative to \mathcal{F} . When $\gamma = L$ is a Legendrian knot in \mathbb{R}^3 , and \mathcal{F} is the framing given by the Seifert surface of L , then $tw(L, \mathcal{F})$ is the Thurston-Bennequin invariant of L denoted by $tb(L)$. [15] When studying Legendrian torus knots, we will sometimes consider the twist with respect to the torus and sometimes with respect to its Seifert surface. Alternatively, the Thurston-Bennequin invariant can be thought of as a self-linking number as follows. Let w be a nonzero vector field along L in $v \cap \xi$ where v is the

normal bundle of L . Let L' be a copy of L defined by pushing L slightly in the w direction. Then $tb(L) = lk(L, L')$.

The rotation number is another invariant of a Legendrian knot. For an oriented Legendrian knot $L \subset \mathbb{R}^3$, let v be a nonzero tangent vector field to L pointing in the direction of the orientation of L . Notice that all of the contact planes project diffeomorphically to the (x, z) -plane and so we have a natural trivialization $w = \frac{\partial}{\partial y}$ of $\xi|_L = L \times \mathbb{R}^2$. The vector field v is in $\xi|_L = L \times \mathbb{R}^2$, and so using this trivialization we can think of v as a path of nonzero vectors in \mathbb{R}^2 . As such, v has a winding number. The rotation number, $r(L)$, is defined as this winding number. Note that if we change the orientation of L , we change the sign of the rotation number.

The invariants $k(L)$, $tb(L)$, and $r(L)$ are called the *classical invariants* of L . In our setting, we will always be considering Legendrian knots and links in a fixed knot type. In [4], Eliashberg and Fraser show that Legendrian unknots are completely determined by their classical invariants, and in [9], Etnyre and Honda show that Legendrian torus knots are completely determined by their classical invariants. In this dissertation, we show that unordered Legendrian links are determined by the classical invariants of their components.

If γ is a Legendrian knot in \mathbb{R}^3 with the standard contact structure $\xi = \ker(dz - ydx)$, then its projection onto the xz -plane is called the *front projection*. In this projection, the Thurston-Bennequin invariant and rotation

number can easily be computed as follows.

$$tb(\gamma) = w(\gamma) - \frac{1}{2}(\#\text{cusps}) \quad (2.1)$$

and

$$r(L) = \frac{1}{2}(D - U) \quad (2.2)$$

where $w(\gamma)$ is the writhe of γ , D is the number of down cusps, and U is the number of up cusps. (See Figures 1.5 and 1.6.) See, for example, [8].

Given a Legendrian knot L it is always possible to produce another knot in the same topological knot type by “stabilization.” If a strand of L in the front project looks like the top of Figure 2.1 then the *stabilization* of L is obtained by removing that strand and replacing it by one of the zig-zags shown at the bottom of Figure 2.1. If down cusps are added, then the stabilization is called *positive* and is denoted by $S_+(L)$. If up cusps are added, then the stabilization is called *negative* and is denoted by $S_-(L)$. Note that

$$tb(S_{\pm}(L)) = tb(L) - 1 \quad (2.3)$$

and

$$r(S_{\pm}(L)) = r(L) \pm 1. \quad (2.4)$$

The order of stabilizations does not matter as stated in the following lemma.

Lemma 2.1.1. *[12] Stabilization is well-defined and $S_+(S_-(K)) = S_-(S_+(K))$.*

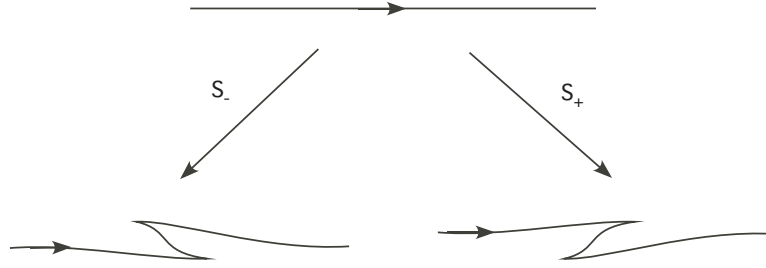


Figure 2.1: Stabilization of Legendrian knots in the front projection.

We will make extensive use of the above lemma in the proofs of the main theorems in Chapters 3 and 4 when we permute the order of stabilizations as needed. For more background and interesting results about stabilizations see, for example, [12].

2.2 Convex Surfaces

As mentioned in Section 2.1, the characteristic foliation is a strong invariant but also very difficult to use in practice. The advantage of convex surface theory is that it reduces the study of characteristic foliations on a surface to the study of multi curves on a surface. These dividing curves, defined below, are more flexible and robust, and encode the necessary information about the contact structure. General background on convex surfaces can be found in [7]. Below we mention some essential definitions and results.

A closed, oriented, embedded surface Σ in a contact manifold (M, ξ) is said to be *convex* if there is a vector field v transverse to Σ whose flow preserves ξ . A closed surface may be isotoped by a C^∞ -small isotopy so that

it is convex ([13]). A convex surface $\Sigma \subset (M, \xi)$ has a naturally associated family of disjointed embedded curves Γ_Σ called the *dividing curves*. More precisely, we define a dividing set Γ_Σ for v to be the set of points x where $v(x) \in \xi(x)$. Intuitively, the dividing set is measuring where the contact planes are “perpendicular” to the convex surface, where the perpendicular direction is given by v . If Σ is clear we will simply write Γ for the dividing set. Γ is a union of smooth curves and arcs which are transverse to the characteristic foliation. If Σ has Legendrian boundary, $\gamma \subset \Gamma$ may be an arc with endpoints on the boundary. The isotopy type of Γ is independent of the choice of v , so we will often call Γ the dividing set of Σ . If the surface undergoes an isotopy staying convex, then these dividing curves change by an isotopy.

We can think of these dividing curves as places where the contact planes “flip.” On each connected complement of the dividing curves, the contact planes will project isomorphically to the tangent planes of the surface with either the same orientation or the opposite orientation. More precisely, $\Sigma \setminus \Gamma = R_+ - R_-$, where R_+ is the subsurface where the orientations of v (from the normal orientation of Σ) and the normal orientation of ξ coincide, and R_- is the subsurface where they are opposite. In $\Sigma \setminus \Gamma$, there will be points p where $\xi_p = T_p \Sigma$. These are singular points of the characteristic foliation given by $\xi \cap T\Sigma$. The singularities may be assumed to be either elliptic or hyperbolic depending on the local degree of the foliation. If Σ is oriented then the singularities have a sign determined by the compatibility of the

orientations of ξ and $T\Sigma$ at the singularities.

The next set of results will allow us to place the torus on which our knot or link sits into convex form, and to deduce much useful information about the associated dividing curves. The first important result is known as Giroux’s Flexibility Theorem. Roughly, this theorem states that it is the isotopy type of the dividing set Γ which dictates the geometry of Σ , not the precise characteristic foliation which is compatible with Γ . This allows us to reduce the classification to a specific characteristic foliation that is compatible with Γ , and we choose a realization of our convex surface which we will call “standard form” later in this section. Before giving the statement of the theorem, we need the following definition. If \mathcal{F} is a singular foliation on Σ , then a disjoint union of properly embedded curves Γ is said to *divide* \mathcal{F} if there exists some I -invariant contact structure ξ on $\Sigma \times I$ such that $\mathcal{F} = \xi|_{\Sigma \times 0}$ and Γ is the dividing set for $\Sigma \times 0$.

Theorem 2.2.1. (Giroux Flexibility, [13]) *Let Σ be a closed convex surface or a compact convex surface with Legendrian boundary, with characteristic foliation $\xi|_{\Sigma}$, contact vector field v , and dividing set Γ . If \mathcal{F} is another singular foliation on Σ divided by Γ , then there is an isotopy ϕ_s , $s \in [0, 1]$, of Σ such that $\phi_0(\Sigma) = \Sigma$, $\xi|_{\phi_1(\Sigma)} = \mathcal{F}$, the isotopy is fixed on Γ , and $\phi_s(\Sigma)$ is transverse to v for all s .*

An isotopy ϕ_s , $s \in [0, 1]$, for which $\phi_s \pitchfork v$ for all s is called *admissible*. The Legendrian Realization Principle (LeRP) below is due to Kanda [16], and the formulation given here is due to Honda [15]. We use this theorem in

the proofs in Chapters 3 and 4 to either make a meridional curve on a torus Legendrian, or to ensure that our link remains Legendrian after an isotopy of the torus.

Theorem 2.2.2. (Legendrian Realization Principle) *Consider a closed curve, C , on a closed convex surface Σ with Legendrian boundary. Assume $C \pitchfork \Gamma_\Sigma$, and every component of $\Sigma \setminus C$ intersects Γ_Σ . Then there exists an admissible isotopy ϕ_s , $s \in [0, 1]$, so that*

- (1) $\phi_0 = id$,
- (2) $\phi_s(\Sigma)$ are all convex,
- (3) $\phi_1(\Gamma_\Sigma) = \Gamma_{\phi_1(\Sigma)}$,
- (4) $\phi_1(C)$ is Legendrian.

The next three results will be used together in order to use the Thurston-Bennequin invariant to determine the number of times a curve (often a component of the link) intersects the dividing curves. That information can then be used to find the twist, which under certain conditions will allow us to make the torus convex while keeping the curve Legendrian. Additionally, the number of times a component of the link intersects the dividing curves can be used to find a “destabilization” of the component. More details on destabilizations will follow.

Theorem 2.2.3. (*Relative Convex Realization Principle*, Kanda [16]). If γ is a Legendrian curve in a surface Σ , then Σ may be isotoped relative to γ so that it is convex if and only if $tw_\Sigma(\gamma) \leq 0$. Moreover, if Σ is convex, then

$$tw_\Sigma(\gamma) = -\frac{1}{2}\#(\gamma \cap \Gamma), \quad (2.5)$$

where Γ is the set of dividing curves for Σ_ξ , and $\#(\gamma \cap \Gamma)$ is the unsigned intersection number.

We will often need to show that a surface with Legendrian boundary can be made convex without moving the boundary. Theorem 2.2 gives us:

Lemma 2.2.1. *Let Σ be a surface with Legendrian boundary. The surface Σ may be made convex if and only if the twist of ξ about each boundary component is less than or equal to zero.*

Note that if Σ has a single Legendrian boundary component $\partial\Sigma = \gamma$, then Σ is a Seifert surface for γ , and so $tw_\Sigma(\gamma) = tb(\gamma)$.

Remark 2.2.4. The framing on the normal bundle of a (p, q) -torus knot K given by its Seifert surface differs from the one induced by the torus T by pq , and so we have $tb(K) - tw(K, T) = pq$. See [8]. \diamond

Lemma 2.2.2. (Kanda [16]). *Suppose Σ has a single boundary component γ , and γ is Legendrian. Then Σ may be made convex if and only if $tb(\gamma) \leq 0$. Moreover, if Σ is convex with dividing curves Γ , then*

$$tb(\gamma) = -\frac{1}{2}\#(\gamma \cap \Gamma) \quad (2.6)$$

and

$$r(\gamma) = \chi(R_+) - \chi(R_-), \quad (2.7)$$

where R_{\pm} are as in the definition of convexity.

In the Introduction chapter, we discussed the idea of adding “zig-zags”, or stabilizations, to a Legendrian knot diagram. In fact, we never actually find “zig-zags” when looking to see if a knot or link has been stabilized. Instead, we locate a bypass as described below.

Definition 2.2.5. Let $\Sigma \subset M$ be a convex surface that is closed or compact with Legendrian boundary. A *bypass* for Σ is an oriented embedded half-disk D with Legendrian boundary, satisfying the following:

- (1) ∂D is the union of two Legendrian arcs a_0 and a_1 which intersect at their endpoints.
- (2) D intersects Σ transversely along a_0 .
- (3) D has the following tangencies along ∂D :
 - (a) positive (negative) elliptic tangencies at the endpoints of $a_0 =$ endpoints of a_1 ,
 - (b) one negative (positive) elliptic tangency on the interior of a_0 , and
 - (c) only positive (negative) tangencies along a_1 , alternating between elliptic and hyperbolic.

- (4) a_0 intersects Γ_Σ at exactly three points, and these three points are the elliptic points of a_0 .

See Figure 2.2 for an illustration of a bypass. We will often call the arc a_1 a bypass for Σ or a bypass for a_0 . The *sign of the bypass* is the sign of the singularity in (b) above.

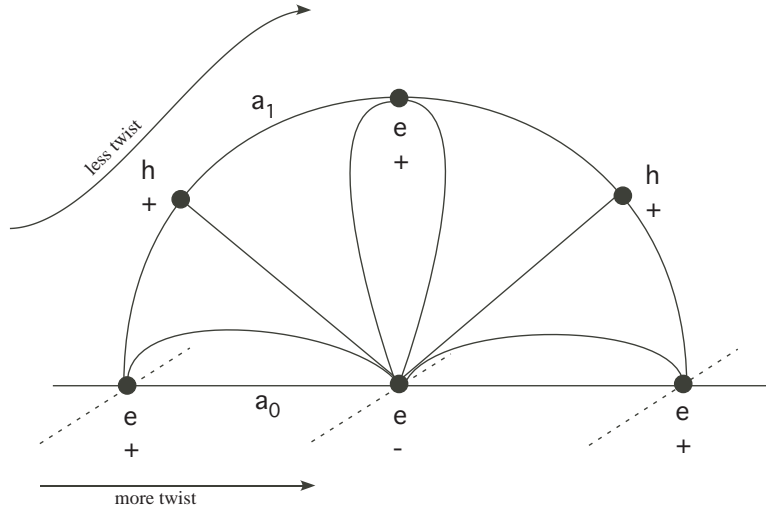


Figure 2.2: A negative bypass. Elliptical singularities are marked with an e and hyperbolic singularities are marked with an h .

There is a clear relationship between the definition of a bypass and a stabilizing disk. Suppose D' is a stabilizing disk for a Legendrian knot γ' , and once stabilized we obtain the knot γ . From the perspective of γ , D' is a bypass, meaning that D' shows how to isotop γ so that it twists less. Moreover, if D is a bypass for a knot γ and γ' is the knot obtained from pushing γ across D , then D may be isotoped so that it is a stabilizing disk for γ' .

In the proofs of the main theorems in Chapters 3 and 4, we will use dividing curves to locate bypasses as follows. Let Σ be a convex surface with Legendrian boundary. It is possible to manipulate the characteristic foliation to make all of the singularities along the boundary (half)-elliptic ([4]). Now if $tb(\partial\Sigma) = -n \leq 0$ then by Lemma 2.2.2 the dividing curves intersect $\partial\Sigma$, $2n$ times. Suppose one of these dividing curves is *boundary parallel*, meaning that it cuts off a half-disk with no other intersections with Γ_Σ . (See Figure 2.3.) We may use the characteristic foliation to flow the dividing curve “away” from the boundary and it will limit to a Legendrian curve α . The curve α will separate off a disk D from Σ which will be a bypass for $\partial\Sigma$.

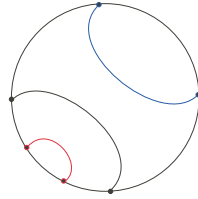


Figure 2.3: In this figure, $tb(\partial\Sigma) = -3$ and so the dividing curves intersect $\partial\Sigma$ four times. The arc shown in blue and red are boundary parallel, and so either of these will determine a bypass for $\partial\Sigma$.

Lemma 2.2.3. ([15]) *Given a boundary-parallel dividing curve δ on a convex surface Σ with Legendrian boundary, one may find a bypass for the boundary, provided that Γ_Σ is not a single arc on $\Sigma = D^2$.*

In many of the proofs in Chapters 3 and 4, we will at some point be moving our link from one torus to a parallel torus by pushing across an

annular region. It is in this annular region that we will find a bypass. If Σ is an annulus, we can use the following proposition to find boundary-parallel dividing curves.

Proposition 2.2.1. (The Imbalance Principle, [15].) *If $\Sigma = S^1 \times [0, 1]$ is convex and has Legendrian boundary where $tw(S^1 \times \{0\}) < tw(S^1 \times \{1\}) \leq 0$, then there exists a boundary parallel dividing curve (and hence a bypass) along $S^1 \times \{0\}$.*

As mentioned earlier, a slight perturbation of a convex surface will not change the isotopy type of the dividing curves. However, moving the surface through a bypass may alter the isotopy type of the dividing curves.

Proposition 2.2.2. ([9]) *Let $A = [0, 1] \times [0, 1]$ be a convex square with three horizontal dividing curves and vertical ruling. Let γ be one of the vertical ruling curves and D a bypass for γ disjoint from A . Then we may isotop A rel boundary by pushing A across D so as to alter the characteristic foliation and dividing curves as shown in Figure 2.4.*

Note that on the torus, the sides of the annulus A are identified, so the number of dividing curves are reduced by 2.

Let T be a convex torus in \mathbb{R}^3 , and assume that ξ is tight. Because ξ is tight, we know that no dividing curve bounds a disk, and hence the dividing curves are parallel homotopically essential curves [15]. If there are $2n$ parallel curves, then using Theorem 2.2.1 we may assume there are $2n$

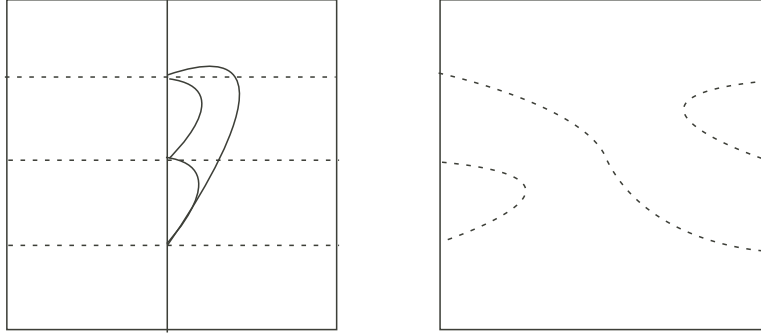


Figure 2.4: Dividing curves on A before and after the isotopy given by the bypass attachment shown.

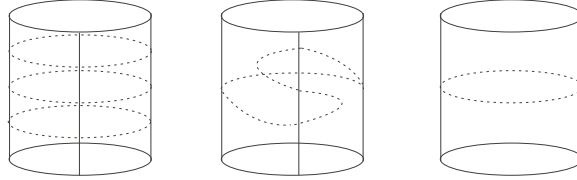


Figure 2.5: The bypass attachment on the torus reduces the number of dividing curves by 2.

curves of singularities in T_ξ , one in each region of the complement of the dividing curves. We call these curves the *Legendrian divides* and their slope the *boundary slope*. Note that these curves will all have the same slope and are Legendrian isotopic (the fact that they are isotopic follows from Lemma 2.2.6 below.) The other leaves in T_ξ form a 1-parameter family of closed curves called the *Legendrian ruling curves*. By Theorem 2.2.1, the slope of the ruling curves can be made to be whatever we want other than the boundary slope. The ruling curves all have the same slope and are clearly isotopic. In Chapters 3 and 4 we will use the fact that a positive torus link can be viewed as a subset of the ruling curves, and a negative torus link

can be viewed as a subset of the Legendrian divides. If the characteristic foliation on a convex torus has this nongeneric form than we say that T_ξ is in *standard form*. (See Figure 2.6.) If $n > 1$ and we can find a bypass for one of the ruling curves, then we can isotop T so as to reduce n by 1. See Figures 2.4 and 2.5. If $n = 1$ then pushing across a bypass changes the slope of the dividing curves.

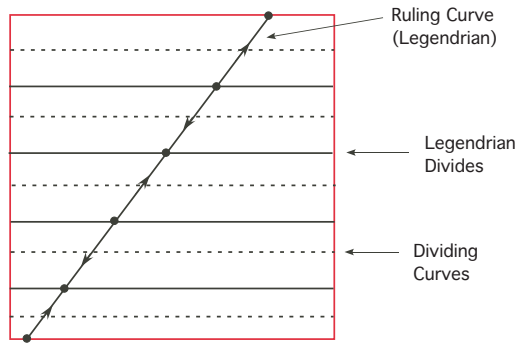


Figure 2.6: Ruling curves, dividing curves, and Legendrian divides in standard form.

Theorem 2.2.6. (Honda [15]) *Let T be a convex torus in standard form with two dividing curves and in some basis for T the slope of the dividing curves is 0. If we can find a bypass on a ruling curve of slope between $-\frac{1}{m}$ and $-\frac{1}{m+1}$, $m \in \mathbb{Z}$, then after pushing T across the bypass the new torus has two dividing curves of slope $-\frac{1}{m+1}$.*

Understanding the classification of Legendrian torus links is closely tied to understanding the contact structures on solid tori. In [16], Kanda proved the following:

Theorem 2.2.7. *For all $q \geq 1$, there is a unique contact structure on $S^1 \times D^2$ with standard convex boundary having two dividing curves of slope $-\frac{1}{q}$.*

The following lemma follows easily from the above results, and is used in the proofs in Chapters 3 and 4 to view our links as sitting on the same convex torus. The proof is given for the reader's convenience.

Lemma 2.2.4. *Let T and T' be two standardly embedded convex tori with exactly 2 dividing curves with slope -1 . Then there exists a contact isotopy from T to T' .*

Proof. Let $V_0 \cup V_1$ and $V'_0 \cup V'_1$ be the Heegard splittings of S^3 associated to the tori T and T' . The slopes of the dividing curves Γ and Γ' are both equal to -1 , so by Theorem 2.2.7 there is a contactomorphism $\phi : V_0 \rightarrow V'_0$. Applying Theorem 2.2.7 again to the other torus, we may extend ϕ to all of S^3 , thus obtaining a contactomorphism of S^3 that takes T to T' . By Theorem 2.1.2, we may find a contact isotopy of S^3 taking T to T' since this theorem gives us that ϕ is in the same path component as the identity. \square

Honda showed that for more general slopes, there will be more choices for the contact structure. However, the contact structure will be uniquely determined if we know some additional information about a convex meridional disk.

Theorem 2.2.8. (Honda [15], Giroux [14]) *Let $p < 0$, $q > 1$. There are $d(p, q) \geq 1$ tight contact structures on $S^1 \times D^2$ with standard convex boundary*

having two Legendrian divides of slope $\frac{q}{p}$. (Here our convention is that the meridian has slope 0.) Moreover, all of these structures are distinguished by the number of positive regions on a convex meridional disk with Legendrian boundary.

The number $d(p, q)$ is determined by the continued fraction decomposition of $\frac{q}{p}$. When studying negative Legendrian torus links, this theorem, along with the fact that $r(\gamma) = \chi(R_+) - \chi(R_-)$ given in Lemma 2.2.2, allows us to determine the contact structure on the torus using information about the rotation number of the components of the link. Furthermore, if we know that we have two dividing curves of a given slope on two different tori, we can use the theorem to find a contactomorphism from one torus to the other.

The following lemma will be used throughout the proofs in Chapters 3 and 4. We will use an annular region between the torus on which our link sits and the parallel torus guaranteed by this theorem to locate a bypass, and hence a destabilization.

Lemma 2.2.5. ([9]) *If $S = D^2 \times S^1$ has convex boundary with boundary slope $s < 0$, then we can find a convex torus parallel to the boundary of S with any boundary slope in $[s, 0)$.*

The lemma below tells us that the Legendrian divides of a given torus can be viewed as the ruling curves of a linearly foliated torus. This lemma is useful as the ruling curves are much more flexible than the Legendrian divides, and can be used to show that the Legendrian divides are isotopic

since the ruling curves are clearly isotopic. The foliation on the new torus will also make it possible to do cyclic permutations of the Legendrian divides.

Lemma 2.2.6. ([9]) *Consider a tight contact structure on $T^2 \times [0, 1]$ with boundary slopes $s_1 = -\frac{1}{m}$ for $T^2 \times \{1\}$ and $s_0 = -\frac{1}{m+1}$ for $T^2 \times \{0\}$, $m \in \mathbb{Z}$. If $s_1 < s < s_0$, then there exists a pre-Lagrangian (= linearly foliated) torus T parallel to $T^2 \times \{i\}$, and every convex torus T' in standard form with slope s is, after contact isotopy, transverse to T , and $T \cap T'$ is exactly the union of the Legendrian divides of T' .*

Let T be a convex standardly embedded torus. We now define an invariant of homology classes of curves on T . The details and proofs of the facts that will follow can be found in [9]. For convenience, we repeat the statements of the facts here and give a brief explanation of their meanings and/or how they will be used in the proof of the theorem in Chapter 4. Given T , let $S^3 = V_0 \cup_T V_1$, where V_0 is the solid torus with meridional curve μ and V_1 is the solid torus with meridional curve λ . From Lemma 2.2.5 we know that inside V_0 there is a solid torus S with two dividing curves of slope $-\frac{1}{m+1}$, where $|p| = mq + e$, and that there is a solid torus S' containing V_0 with two dividing curves of slope $-\frac{1}{m}$. Let $T_m = \partial S'$ and $T_{m+1} = \partial S$. Let w be a section of $\xi|_T$ that is transverse to and twists along the Legendrian ruling curves and is tangent to the Legendrian divides. Let v be any non-zero section of ξ . If γ is a closed oriented curve on T , then set $f_T(\gamma)$ equal to the rotation of v relative to w along γ . We similarly define $f_m(\gamma)$ and $f_{m+1}(\gamma)$ for curves on T_m and T_{m+1} . Note that if γ is an (r, s) -ruling curve

or Legendrian divide, then $f_T(\gamma) = r(\gamma)$. In [6], it is shown that f_T is well-defined on homology classes and that f_T is unchanged if we isotop T among convex tori in standard form. The facts that we will use concerning these invariants are:

1. $f_T(\mu) = 1 - q$ or $q - 1$
2. $f_T(\lambda) \in \{m - 1, m - 3, \dots, 1 - m\}$
3. If $f_T(\mu) = 1 - q$ then $f_T(\lambda) = f_m(\lambda) + (m - |p|)$ So $f_T(\lambda) \in \{2m - |p| - 1, 2m - |p| - 3, \dots, 1 - |p|\}$
4. If $f_T(\mu) = q - 1$ then $f_T(\lambda) = f_m(\lambda) + (|p| - m)$ So $f_T(\lambda) \in \{|p| - 1, |p| - 3, \dots, |p| - 2m + 1\}$

From the above properties we know that

$$r(K) = pf_T(\mu) + qf_T(\lambda).$$

Thus the possible values for $r(K)$ lie in

$$\{\pm(|p| - q - 2nq) | 0 \leq n \leq m - 1\}.$$

The important things to understand about the above facts are as follows. The first fact comes from Equation (2.7) and from the possible configurations of the dividing curves. (See Figure 2.7.) V_m is a standard neighborhood of an unknot with $tb = -m$. There are m of these with rotation numbers

$m-1, m-3, \dots, 1-m$. In this way, we can think of $f_T(\lambda)$ as determining V_m . If we know the rotation number of the knot and the value of $f_T(\mu)$, then this will determine $f_T(\lambda)$, which in turn will give us the value of $f_m(\lambda)$. Once we know $f_m(\lambda)$, we have determined V_m , i.e. we have determined the torus on which K sits. Facts (3) and (4) are combined in the proof of the theorem in Chapter 4 in order to determine the tori on which a link sits when all components have maximal Thurston-Bennequin invariant and their rotation numbers differ by $2e$ or $2(q-e)$. See the proof of Lemma 4.0.6 for details.

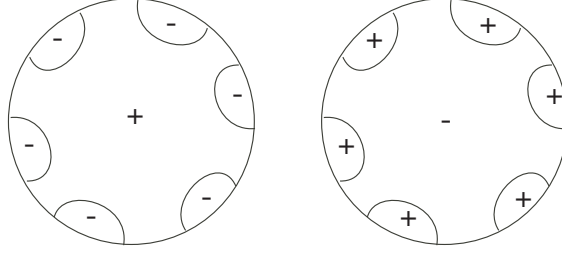


Figure 2.7: Possible configurations of the dividing curves that give that $f_T(\mu) = 1 - q$ or $q - 1$.

2.3 Algebraic Invariants

In order to show that certain permutations are not possible for negative torus links, we will use the algebraic methods detailed in Mishachev's paper [17]. We assume that the reader is familiar with the Chekanov-Eliashberg *DGA* associated to a Legendrian link [1]. The relevant ideas introduced by Mishachev will be repeated here.

Very briefly, a *DGA* is a free, noncommutative, unital, differential graded algebra. The generators of the algebra A are the Reeb chords of the Legendrian link; each generator has a grading given by its Conley-Zehnder index; a differential $\partial : A \rightarrow A$ is obtained by counting particular holomorphic curves in the symplectization of the contact manifold. See, for example, [10]. The *DGA* can easily be calculated from a generic front projection of a Legendrian link as follows. Generators of the algebra are the crossings and right cusps, and the differential (∂) calculation is given by counting immersed disks. As the link undergoes an isotopy, the front projection will change and so the algebra will change. Additional generators may be introduced during the isotopy, which introduces the notion of a “stabilization” of a *DGA*.

Definition 2.3.1. If A is a differential graded algebra, the *stabilization* of A , SA , is the free product $A \amalg S$, where $S = \mathbb{Z}_2\langle a, b \rangle$ and $\partial(a) = b$, $\partial(b) = 0$.

Chekanov proved the following:

Theorem 2.3.2. ([1]) *Differential algebras of Legendrian isotopic links are stable isomorphic, that is, $S^n A = S^m A'$ for some m, n .*

The above theorem tells us that the stable isomorphism class of the algebra associated to a generic projection of the Legendrian link is an invariant of the link. A goal is to find computable invariants from the stable isomorphism class of the algebra.

As mentioned in the introduction, Mishachev was able to extract computable invariants from the algebra to study permutations of the “n-copy”

of a Legendrian unknot. Recall that the n -copy of a Legendrian knot is obtained by taking n identical copies of the knot and translating each copy slightly in the transverse direction so that they are all disjoint. If the original knot has l left cusps, the n -copy will have nl left cusps that will alternate between the n components l times. For example, if we color the three components of a 3-copy of a knot with 2 left cusps black, red, and blue; the cusps would alternate black, red, blue, black, red, blue. In order to prove what permutations of the n -copy of a Legendrian knot K are possible, we will make use of the following result.

Proposition 2.3.1. ([17] Proposition 4.1a) *For any Legendrian link L either 1) All permutations of the n -copy of L are possible by Legendrian isotopy, and that is when all permutations of the 3-copy are possible, or 2) Only cyclic permutations are possible, and that is when 12 to 21 is possible and 123 to 132 is not, or 3) No permutations are possible.*

To study permutations of links, Mishachev introduced the notion of “link degree” to generators of the DGA . Roughly, the *link degree* records which strand is the overcrossing and which is the undercrossing. In Figure 5.2, vertex $b_{m,n}$ has degree $[1 \rightarrow 3]$, vertex $c_{m,n}$ has degree $[3 \rightarrow 2]$ and vertex $b_{m,n}c_{m,n}$ has degree $[1 \rightarrow 2]$. We will only be interested in the link degree of generators coming from certain crossings of the link, and not from the cusps. The link degree induces a splitting of the algebra as given in the following theorem.

Theorem 2.3.3. (Splitting Theorem, [17]) *The differential algebra A splits, $A = \bigoplus_{g \in G_N} A_g$, where $G_N = \pi_1(\mathbb{R}^3/L)$. The differential ∂ preserves this splitting. Algebras of isotopic Legendrian links are componentwise stable isomorphic,*

$$\bigoplus_{g \in G_N} S^n A_g = \bigoplus_{g \in G_N} S^n A'_g.$$

In particular, a permutation σ of components of the Legendrian link by Legendrian isotopy induces a componentwise automorphism,

$$\bigoplus_{g \in G_N} S^n A_g \rightarrow \bigoplus_{\sigma(g) \in G_N} S^n A_{\sigma(g)}.$$

Note that for a link L , the group $G_N = \pi_1(\mathbb{R}^3/L)$ is a free group on $N - 1$ variables, where N is the number of components of the link. [17]

Recall that an augmentation ϵ of A is a ring homomorphism $\epsilon : A \rightarrow \mathbb{Z}_2$ such that $\epsilon \circ \partial = 0$ and $\epsilon(1) = 1$. (See [1], [11], [19] for more information on augmentations.) Augmentations allow us to get a handle on computable invariants. For example, it is possible to form polynomial invariants as in [18]. An augmentation ϵ is called proper if $\epsilon(A_g) = 0$ for $g \neq 1$. This means that we augment only vertices with link degree $[i \rightarrow i]$. Proper augmentations of a link are in one-to-one correspondence with augmentations of its knot components [17]. Given the homomorphism $\epsilon : A \rightarrow \mathbb{Z}_2$, ϵ is completely determined by its values on the generators of A , i.e. all values $\epsilon(a_i)$ for $a_i \in A$.

There is an induced automorphism of the algebra, ϕ_ϵ , which is a change in the coordinate system, via $\phi_\epsilon(a_i) = a_i + \epsilon(a_i)$. Then ϵ is an augmentation if in the new coordinate system ∂ has no units (constant terms). For example, if initially for some $a, b, c \in A$ we have

$$\partial(a) = 1 + bc$$

and $\epsilon(a) = 0$, $\epsilon(b) = \epsilon(c) = 1$, then the new boundary map, ∂^ϵ , induced by ϵ is given by

$$\partial^\epsilon = \phi_\epsilon \circ \partial \circ \phi_\epsilon^{-1}$$

and satisfies

$$\partial^\epsilon(a) = 1 + (b+1)(c+1) = 1 + bc + c + b + 1 = bc + c + b.$$

If A has an augmentation, then SA has an augmentation $S\epsilon$. This new augmentation is explained in detail below.

Lemma 2.3.1. *If $\phi : A \rightarrow A$ is any differential preserving automorphism of A such that for some augmentations ϵ_1, ϵ_2 of A we have $\epsilon_1 = \epsilon_2 \circ \phi$, then*

$$\phi_{\epsilon_2} = \phi \circ \phi_{\epsilon_1} \circ \phi^{-1}.$$

Thus

$$\partial^{\epsilon_1} = \phi \circ \partial^{\epsilon_2} \circ \phi^{-1},$$

and so $\partial^{\epsilon_2} \circ \phi = \phi \circ \partial^{\epsilon_1}$.

Proof. For any $a \in A$, we have that $\phi_{\epsilon_1}(a) = a + \epsilon_1(a)$, where $\epsilon_1(a) = 0$ or $\epsilon_1(a) = 1$. Then

$$\begin{aligned} \phi^{-1} \circ \phi_{\epsilon_2} \circ \phi(a) &= \phi^{-1}(\phi(a) + \epsilon_2(\phi(a))) \\ &= a + \phi^{-1}(\epsilon_2(\phi(a))) \\ &= a + \epsilon_1(a). \end{aligned}$$

The last equality holds as $\phi^{-1}(0) = 0$ and $\phi^{-1}(1) = 1$. Thus $\phi_{\epsilon_2} = \phi \circ \phi_{\epsilon_1} \circ \phi^{-1}$.

Now recall that $\partial^{\epsilon_1} = \phi_{\epsilon_1} \circ \partial \circ \phi_{\epsilon_1}^{-1}$ and $\partial^{\epsilon_2} = \phi_{\epsilon_2} \circ \partial \circ \phi_{\epsilon_2}^{-1}$. Since ϕ is a differential preserving automorphism, we have that $\phi^{-1} \circ \partial \circ \phi = \partial$. We then have

$$\begin{aligned} \partial^{\epsilon_2} &= \phi_{\epsilon_2} \circ \partial \circ \phi_{\epsilon_2}^{-1} \\ &= \phi \circ \phi_{\epsilon_1} \circ \phi^{-1} \circ \partial \circ \phi \circ \phi_{\epsilon_1}^{-1} \circ \phi^{-1} \\ &= \phi \circ \phi_{\epsilon_1} \circ \partial \circ \phi_{\epsilon_1}^{-1} \circ \phi^{-1} \\ &= \phi \circ \partial^{\epsilon_1} \circ \phi^{-1}, \end{aligned}$$

and so $\partial^{\epsilon_2} \circ \phi = \phi \circ \partial^{\epsilon_1}$. □

Note that not all Legendrian torus links will have associated algebras with a proper augmentation, so the following proposition makes it useful to work with a 2-copy of the knot.

Proposition 2.3.2. (*[17]*) *An algebra A associated to a fixed projection of the 2-copy of any Legendrian link has at least one augmentation.*

Consider a differential algebra A with a fixed proper augmentation. Given this augmentation, Mishachev defines \mathbb{T}_g for $g = [i \rightarrow j]$, denoted as \mathbb{T}_{ij} , as a quotient of A . \mathbb{T}_{ij} can be viewed as vector space spanned by vertices with link degree $[i \rightarrow j]$. For example, in Figure 5.2, \mathbb{T}_{13} is spanned by all of the $b_{i,j}$'s. We will also focus on what Mishachev calls the ijk -localization for A . Let $\gamma = \{[i \rightarrow j], [j \rightarrow k], [i \rightarrow k]\}$. The ijk -localization for A is the space \mathbb{T}_{ijk} generated by the generators of A with link degrees from γ . The only quadratic elements are products of generators of degree $[i \rightarrow j]$ and $[j \rightarrow k]$. As a \mathbb{Z}_2 -vector space, \mathbb{T}_{ijk} is isomorphic to $\mathbb{T}_{ij} \oplus \mathbb{T}_{jk} \oplus \mathbb{T}_{ik} \oplus (\mathbb{T}_{ij} \otimes \mathbb{T}_{jk})$. The differential $(\partial_{ijk}^\epsilon)$ is obtained from ∂^ϵ by erasing all nonlinear terms except products ab with $g(a) = [i \rightarrow j], g(b) = [j \rightarrow k]$. In essence this means that we are looking at many fewer generators, and we only see particular quadratic terms. For example, in Chapter 5 we examine the link $6(-4, 3)$. The full algebra associated to this link has 374 generators, but T_{ijk} has only 144 generators. In Figure 5.2, the marked vertices are the only ones that we need in \mathbb{T}_{132} . The only quadratic terms here will have link degree $[1 \rightarrow 2]$.

Definition 2.3.4. Given an augmentation ϵ of A , its associated *characteristic ijk -algebra* is $CH_{ijk}(\epsilon) = \mathbb{T}_{ijk}/Im(\partial^\epsilon)_{ijk}$.

When the augmentation ϵ is clear, we will write ∂_{ijk} and CH_{ijk} for $(\partial^\epsilon)_{ijk}$ and $CH_{ijk}(\epsilon)$.

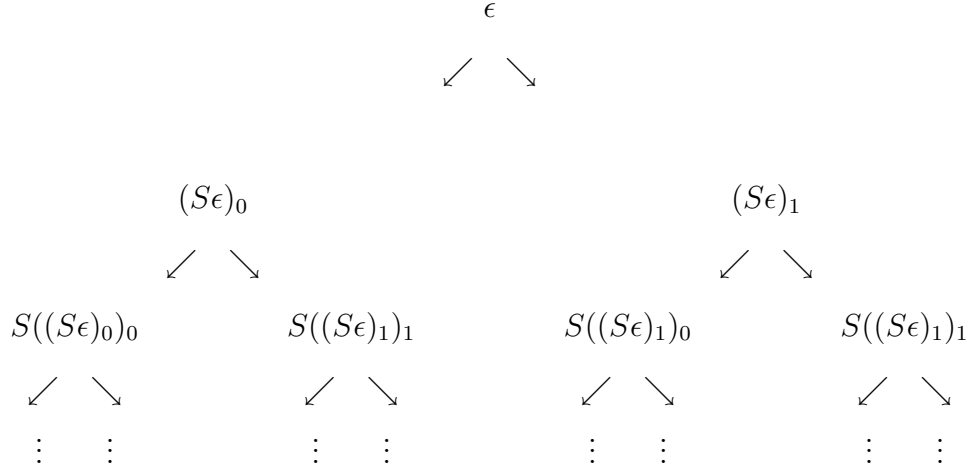
Our strategy will be to distinguish different orderings of a link by studying zero divisors of $CH_{ijk}(\epsilon)$. A zero divisor in $CH_{ijk}(\epsilon)$ is a product $ab \in Im(\partial^\epsilon)_{ijk}$ where $a, b \notin Im(\partial^\epsilon)_{ijk}$. By construction, ab is in the image of

$Im(\partial^\epsilon)_{ijk}$ only if a has degree $[i \rightarrow j]$ and b has degree $[j \rightarrow k]$, so ab has degree $[i \rightarrow k]$.

As mentioned earlier, the algebra of a Legendrian link is only defined up to stabilization. Note that the stabilization SA induces a stabilization $S\mathbb{T}_{ijk}$. We then get a corresponding stabilization of the characteristic ijk -algebra. Since CH_{ijk} depends on an augmentation of A , we first need to understand the augmentations of SA before we can define the stabilization of CH_{ijk} .

Definition 2.3.5. Given an augmentation of A , $\epsilon : A \rightarrow \mathbb{Z}_2$, a *stabilization* of ϵ is an augmentation $S\epsilon : SA \rightarrow \mathbb{Z}_2$, where $S\epsilon|_A = \epsilon$. More generally, an n -fold stabilization of ϵ is an augmentation $S^n\epsilon : S^n A \rightarrow \mathbb{Z}_2$, where $S^n\epsilon|_A = \epsilon$.

For every augmentation ϵ of an algebra A , there are in fact two augmentations of the stabilization $SA = A \amalg \mathbb{Z}_2\langle a, b \rangle$. The first, $(S\epsilon)_0$ is defined by $\epsilon(a) = \epsilon(b) = 0$. The second, $(S\epsilon)_1$, is defined by $\epsilon(a) = 1$, $\epsilon(b) = 0$. It is clear that these are the only two choices for $\epsilon(a)$ and $\epsilon(b)$ since $(S\epsilon)_0|_A = (S\epsilon)_1|_A = \epsilon$, and $\partial(a) = b$, $\partial(b) = 0$. More generally, $S^n A$ will have 2^n augmentations determined by the two choices of $\epsilon(a_i)$ at each stage of stabilization of the algebra. For example, $S^2 A$ has $2^2 = 4$ augmentations: $S((S\epsilon)_0)_0$, $S((S\epsilon)_1)_1$, $S((S\epsilon)_1)_0$, and $S((S\epsilon)_0)_1$. All of the stabilizations of a fixed augmentation ϵ can be viewed by a diagram like the one below.



Definition 2.3.6. Two augmentations ϵ and ϵ' of A are called *stably equivalent* if there exist n -fold stabilizations $S^n\epsilon$, $S^n\epsilon'$ of ϵ , ϵ' , and a differential preserving automorphism $\alpha : S^n A \rightarrow S^n A$ such that the diagram

$$\begin{array}{ccc}
 S^n A & \xrightarrow{\alpha} & S^n A \\
 S^n \epsilon' \searrow & & \swarrow S^n \epsilon \\
 & \mathbb{Z}_2 &
 \end{array}$$

commutes. In other words, $S^n \epsilon' = S^n \epsilon \circ \alpha$.

Definition 2.3.7. For a fixed algebra A , define an equivalence relation on the set of augmentations of A by $\epsilon \sim \epsilon'$ if ϵ and ϵ' are stably equivalent.

A set of equivalent augmentations of A forms an *augmentation class*. We will denote the augmentation class of ϵ by $[\epsilon]$.

Lemma 2.3.2. *If $(S\epsilon)_0$ and $(S\epsilon)_1$ are the two stabilizations of ϵ , then $(S\epsilon)_0 \sim (S\epsilon)_1$.*

Proof. Recall that $(S\epsilon)_0$ is given by $\epsilon(a_1) = \epsilon(b_1) = 0$, and $(S\epsilon)_1$ is given by $\epsilon(a_1) = 1$, $\epsilon(b_1) = 0$. Let $S^2A = A \amalg \mathbb{Z}_2\langle a_1, b_1 \rangle \amalg \mathbb{Z}_2\langle a_2, b_2 \rangle$. Let $\alpha : S^2A \rightarrow S^2A$ be the differential preserving automorphism given by $\alpha|_A = id$, $\alpha(a_1) = a_2$, $\alpha(a_2) = a_1$, $\alpha(b_1) = b_2$, and $\alpha(b_2) = b_1$. Then

$$S((S\epsilon)_0)_1 = S((S\epsilon)_1)_0 \circ \alpha,$$

and so $(S\epsilon)_0 \sim (S\epsilon)_1$. □

It now makes sense to talk about *the* stabilization of an augmentation ϵ of A , i.e. the diagram before Definition 2.3.6 representing all of the stabilizations of ϵ can be condensed into the following diagram of augmentation classes of stabilizations of ϵ .

$$\begin{array}{c} \epsilon \\ \downarrow \\ [S\epsilon] \\ \downarrow \\ [S^2\epsilon] \\ \downarrow \\ \vdots \end{array}$$

In practice, we will suppress the bracket notation and simply write $S\epsilon$ to mean any representative of $[S\epsilon]$

Given a DGA , A , and an augmentation ϵ of A , the stabilization of the characteristic algebra $CH_{ijk}(\epsilon)$ of A is denoted by $S(CH_{ijk})(S\epsilon)$ and is the characteristic algebra of SA . Recall that SA is the addition of two new generators, say a and b , to A . Since ∂ preserves the link grading, either both a and b are in \mathbb{T}_{ijk} or neither are. If neither, then $S(CH_{ijk})(S\epsilon) = CH_{ijk}(\epsilon)$. If both, then only one of a, b is not in the image of ∂ , and so $S(CH_{ijk})$ is the addition of one new generator to CH_{ijk} of any appropriate link degree. Since the single generator added to form $S(CH_{ijk})$ does not appear in the $\partial^{S\epsilon}$ calculation of any other generator, we have the following result.

Proposition 2.3.3. (*[17]*) *Both $CH_{ijk}(\epsilon)$ and $S(CH_{ijk})(S\epsilon)$ have divisors of zero or neither have.*

In addition, as the A undergoes an isomorphism, the corresponding characteristic algebras will both have zero divisors or neither will. This can be proved using an argument similar to the proofs of Lemmas 2.3.3 and 2.3.5. The above proposition tells us that given an algebra A associated to a fixed projection of a link, \bar{L} , and a unique proper augmentation of A , whether or not the associated algebra CH_{ijk} has divisors of zero is an invariant of A , and hence of \bar{L} , and in fact of the link L . In Mishachev's work with unknots (i.e. (np, n) -torus knots with $p < 0$), there is a unique proper augmentation of the associated algebra, and so he was able to use Proposition 2.3.3 to distinguish the different orderings of the links of unknots. For $(np, n2)$ -torus knots, $p < 0$ and $(p, 2) = 1$, this is still the case. Unfortunately, for $q > 2$ there are many proper augmentations of the algebra associated to a negative

(p, q) -torus knot. The following lemmas will allow us to define an invariant of CH_{ijk} , and hence of the link, by counting augmentation classes that yield divisors of zero.

Lemma 2.3.3. *Let ϵ_1 and ϵ_2 be two augmentations of A . If ϵ_1 and ϵ_2 are stably equivalent, i.e. $[\epsilon_1] = [\epsilon_2]$, then for all ijk , $CH_{ijk}(\epsilon_1)$ has divisors of zero if and only if $CH_{ijk}(\epsilon_2)$ has divisors of zero.*

Proof. By Proposition 2.3.3, we know that if $CH_{ijk}(\epsilon_1)$ has zero divisors if and only if $S^n CH_{ijk}(S^n \epsilon_1)$ has zero divisors. Since ϵ_1 is equivalent to ϵ_2 , by definition there exists a differential preserving automorphism $\alpha : S^n A \rightarrow S^n A$ such that $S^n \epsilon_2 = S^n \epsilon_1 \circ \alpha$. Applying Lemma 2.3.1, we have $\partial^{S^n \epsilon_2} \circ \alpha = \alpha \circ \partial^{S^n \epsilon_1}$. Now suppose ab is a zero divisor of $S^n CH_{ijk}(S^n \epsilon_1)$, meaning $ab \in \text{Im}(\partial^{S^n \epsilon_1})$, but $a, b \notin \text{Im}(\partial^{S^n \epsilon_1})$. That is, there exists some $x \in A$ such that $ab = \partial^{S^n \epsilon_1}(x)$. Then

$$\partial^{S^n \epsilon_2} \circ \alpha(x) = \alpha \circ \partial^{S^n \epsilon_1}(x) = \alpha(ab) = \alpha(a)\alpha(b),$$

and so $\alpha(a)\alpha(b) \in \text{Im}(\partial^{S^n \epsilon_2})$, but $\alpha(a), \alpha(b) \notin \text{Im}(\partial^{S^n \epsilon_2})$ as this would otherwise imply $a, b \in \text{Im}(\partial^{S^n \epsilon_1})$. Thus $S^n CH_{ijk}(S^n \epsilon_2)$ has zero divisors. Applying Proposition 2.3.3 again tells us that $S^n(CH_{ijk}(S^n \epsilon_2))$ has zero divisors if and only if $CH_{ijk}(\epsilon_2)$ has zero divisors. Thus $CH_{ijk}(\epsilon_1)$ has divisors of zero if and only if $CH_{ijk}(\epsilon_2)$ has divisors of zero. \square

Remark 2.3.8. Lemma 2.3.3 gives us a way to show that two augmentations are not equivalent. In Chapter 5, we will have two augmentations ϵ_1 and ϵ_2

such that $CH_{123}(\epsilon_1)$ and $CH_{123}(\epsilon_2)$ both have divisors of zero, but $CH_{132}(\epsilon_1)$ has divisors of zero while $CH_{132}(\epsilon_2)$ does not. Thus ϵ_1 and ϵ_2 cannot be equivalent. \diamond

The above lemma tells us that all elements of an augmentation class have zero divisors or do not. We next show that a differential preserving automorphism of the algebra permutes the augmentation classes of the algebra.

Let $\phi : A \rightarrow A$ be a differential preserving automorphism of A . For any augmentation ϵ of A , there is an induced augmentation ϵ' of $\phi(A) = A$ given by $\epsilon' = \epsilon \circ \phi^{-1}$. We will sometimes denote ϵ' by $\phi(\epsilon)$.

Lemma 2.3.4. *Let $\phi : A \rightarrow A$ be a differential preserving automorphism of A . If ϵ_1 and ϵ_2 are two augmentations of A then $\epsilon_1 \sim \epsilon_2$ if and only if $\phi(\epsilon_1) \sim \phi(\epsilon_2)$. It follows that ϕ induces a permutation on the augmentation classes of A .*

Proof. If $\epsilon_1 \sim \epsilon_2$, then by definition there exists a differential preserving automorphism $\alpha : S^n A \rightarrow S^n A$ such that $S^n \epsilon_1 = S^n \epsilon_2 \circ \alpha$. The stabilization of $\phi(\epsilon) = \epsilon \circ \phi^{-1}$ is $S^n \epsilon \circ (S\phi)^{-1}$ where $S\phi : SA \rightarrow SA$, $S\phi|_A = \phi$, and $S\phi$ acts as the identity on $\mathbb{Z}_2\langle a, b \rangle$. Let $\beta = S^n \phi \circ \alpha \circ (S^n \phi)^{-1}$. Then β is a differential preserving automorphism of $S^n A$. Since $S^n \epsilon_1 = S^n \epsilon_2 \circ \alpha$, we have

$$S^n \epsilon_1 \circ (S^n \phi)^{-1} = S^n \epsilon_2 \circ \alpha \circ (S^n \phi)^{-1} = S^n \epsilon_2 \circ (S^n \phi)^{-1} \circ \beta.$$

Thus $\epsilon_1 \circ \phi^{-1} \sim \epsilon_2 \circ \phi^{-1}$, i.e. $\phi(\epsilon_1) \sim \phi(\epsilon_2)$. This tells us that ϕ takes augmentation classes to augmentation classes, so that $\phi([\epsilon]) = [\phi(\epsilon)]$ is well-

defined. Applying the same argument to ϕ^{-1} , we get an induced map on equivalence classes $\phi^{-1}([\epsilon]) = [\phi^{-1}(\epsilon)]$. Note that $\phi^{-1} \circ \phi = id$ and $\phi \circ \phi^{-1} = id$, so ϕ induces a permutation on the augmentation classes of A . \square

Remark 2.3.9. The correspondence of augmentation classes described in the above lemma holds at all levels of stabilization of the algebra. That is, ϕ induces a permutation on the augmentation classes of A if and only if ϕ induces a permutation on the augmentation classes of $S^n A$ for all n . This follows from the definition of a stabilization of an augmentation and from Proposition 2.3.3. \diamond

Lastly, we show that zero divisors of CH_{ijk} are preserved under an automorphism of A that induces an isomorphism \mathbb{T}_{123} to \mathbb{T}_{132} .

Lemma 2.3.5. *If $\phi : A \rightarrow A$ is a differential preserving automorphism of A that induces an isomorphism \mathbb{T}_{123} to \mathbb{T}_{132} , and ϵ is any augmentation of A , then $CH_{123}(\epsilon)$ has divisors of zero if and only if $CH_{132}(\phi(\epsilon))$ has divisors of zero.*

Note that the above statement is well-defined as we know by Lemma 2.3.3 that whether or not CH_{ijk} has zero divisors does not depend on the choice of ϵ from a given augmentation class.

Proof. If we write the augmentation $\phi(\epsilon) = \epsilon' = \epsilon \circ \phi^{-1}$ as $\epsilon = \epsilon' \circ \phi$ then we can apply Lemma 2.3.1 to get

$$(\partial^{\epsilon'})_{132} = \phi \circ \partial_{123}^{\epsilon} \circ \phi^{-1}.$$

Now suppose ab is a zero divisor of $CH_{123}(\epsilon)$, meaning $ab \in \text{Im}(\partial_{123}^\epsilon)$, but $a, b \notin \text{Im}(\partial_{123}^\epsilon)$. That is, there exists some $x \in A$ such that $ab = \partial_{123}^\epsilon(x)$. Then

$$\partial_{132}^{\epsilon'} \circ \phi(x) = \phi \circ \partial_{123}^\epsilon(x) = \phi(ab) = \phi(a)\phi(b),$$

and so $\phi(a)\phi(b) \in \text{Im}(\partial_{132}^{\epsilon'})$, but $\phi(a), \phi(b) \notin \text{Im}(\partial_{132}^{\epsilon'})$ as this would otherwise imply $a, b \in \text{Im}(\partial_{123}^\epsilon)$. Thus $CH_{123}(\epsilon)$ has divisors of zero if and only if $CH_{132}(\epsilon')$ has divisors of zero. \square

Let L denote the unordered 3-copy of K , and L_{123} and L_{132} denote L with different orderings. Given a projection of a link, \bar{L} , its associated algebra $A_{\bar{L}}$, and a unique proper augmentation of $A_{\bar{L}}$, whether or not the associated algebra CH_{ijk} has divisors of zero is an invariant of \bar{L} . When $A_{\bar{L}}$ has many augmentations, it is not enough to show that for any fixed augmentation ϵ , $CH_{123}(\epsilon)$ has divisors of zero while $CH_{132}(\epsilon)$ does not. We define a number z_{ijk} that counts the number of augmentation classes that yield zero divisors in CH_{ijk} . We will show below that in order to prove that two links are different, it suffices to show that for one of the links there exists zero divisors for a fixed number of the possible augmentation classes, while for the other link there exists zero divisors for a different number of the possible augmentation classes.

Definition 2.3.10. For $i \in \{123, 132\}$, let z_i equal the number of augmentation classes $[\epsilon]$ of $A_{\bar{L}}$ such that for any ϵ in $[\epsilon]$, $CH_i(\epsilon)$ has zero divisors.

We know that z_i is well-defined by Lemma 2.3.3 since this lemma tells us

that $CH_i(\epsilon)$ has zero divisors or not for all ϵ in $[\epsilon]$. Note that if we choose two different elements of the augmentation class the associated characteristic algebras will be different, but they will both have zero divisors or both will not.

Lemma 2.3.6. *If $L_{123} = L_{132}$ then for any fixed projection \bar{L} of L , $z_{123} = z_{132}$*

Proof. Let $A_{\bar{L}}$ be the algebra associated to a fixed projection \bar{L} of L . Label the equivalence classes of the proper augmentations of $A_{\bar{L}}$ as $[\epsilon_1], \dots, [\epsilon_n]$. Let $\epsilon_1, \dots, \epsilon_n$ be representatives of each augmentation class. We then have algebras $CH_{123}(\epsilon_1), \dots, CH_{123}(\epsilon_n)$ associated to L_{123} . If $L_{123} = L_{132}$ then there is a loop of Legendrian links starting and ending at \bar{L} , a corresponding differential preserving automorphism ϕ of some stabilization of A , $S^m A$. In addition, by the Splitting Theorem, Theorem 2.3.3, there is an isomorphism taking $S^m(\mathbb{T}_{123})$ to $S^m(\mathbb{T}_{132})$. Note that $[S^m \epsilon_1], \dots, [S^m \epsilon_n]$ are all distinct. By Lemma 2.3.4, ϕ permutes the augmentation classes of $S^m A$. As noted in Remark 2.3.9, we then get an induced permutation σ of the augmentations of $A_{\bar{L}}$, meaning for all i , $\phi([\epsilon_i]) = [\epsilon_{\sigma(i)}]$. Thus, the characteristic algebras associated to L_{132} are $CH_{132}(\epsilon_{\sigma(1)}), \dots, CH_{132}(\epsilon_{\sigma(n)})$. By Lemma 2.3.5, $S^m CH_{123}(S^m \epsilon_i)$ has zero divisors if and only if $S^m CH_{132}(\phi(S^m \epsilon_i))$ has zero divisors. Proposition 2.3.3 tells us that both $CH_{ijk}(\epsilon_i)$ and $S^m(CH_{ijk}(\epsilon_i))$ have zero divisors or both do not. We now have that the sets

$$\{CH_{123}(\epsilon_1), \dots, CH_{123}(\epsilon_n)\} \quad \text{and} \quad \{CH_{132}(\epsilon_{\sigma(1)}), \dots, CH_{132}(\epsilon_{\sigma(n)})\}$$

have the same number of elements with zero divisors. Therefore, $z_{123} = z_{132}$. \square

Chapter 3

Positive Torus Links

In this chapter, we will classify all (np, nq) -torus links with p and q relatively prime and $p > q > 0$. We know from the work of Etnyre and Honda in [9] that there is a unique positive (p, q) -torus knot with maximal Thurston-Bennequin invariant equal to $pq - p - q$. The rotation number of this knot is equal to 0. These torus knots with maximal Thurston-Bennequin invariant will have the form shown in Figure 3.1. All other (p, q) -torus knots are stabilizations of the unique one with maximal Thurston-Bennequin invariant. This relationship can be represented by a tree diagram like the one in Figure 3.3. The unordered classification of positive torus links is given by the following theorem.

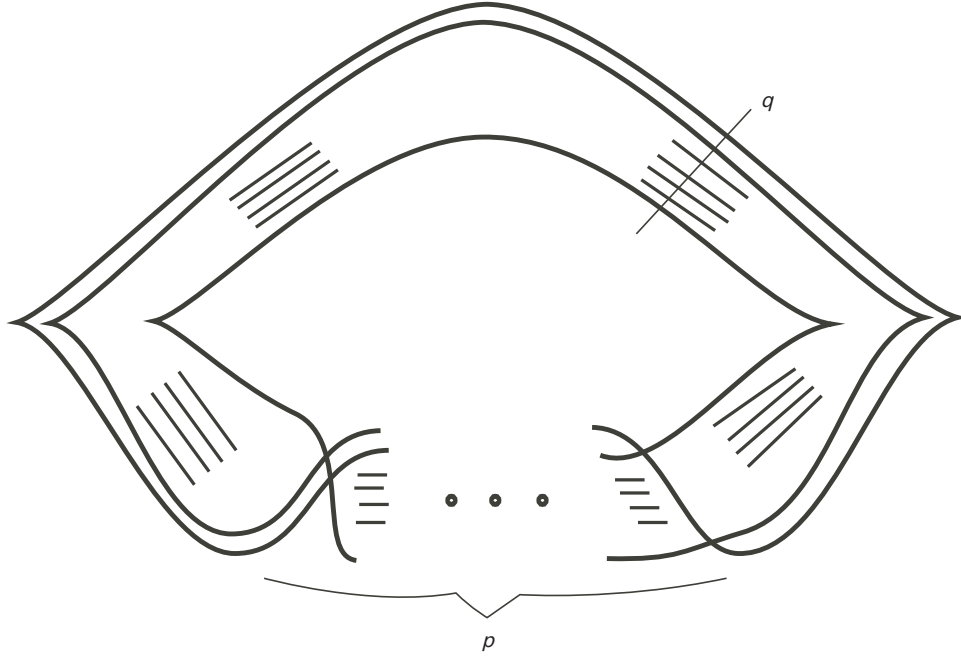


Figure 3.1: Positive Legendrian torus knots.

Theorem 3.0.11. (Unordered Positive Torus Link Classification) *Let L be an unordered Legendrian (np, nq) -torus link with components K_1, \dots, K_n , and L' be an unordered Legendrian (np, nq) -torus link with components K'_1, \dots, K'_n . Then L and L' are Legendrian isotopic if and only if there exists a permutation σ of $\{1, 2, \dots, n\}$ such that $tb(K_i) = tb(K'_{\sigma(i)})$ and $r(K_i) = r(K'_{\sigma(i)})$. Moreover, the components of an (np, nq) -torus link can be any (p, q) torus knots.*

Looking at the tree diagram in Figure 3.3, the above theorem tells us that any n vertices, allowing repeats, will make an $(n3, n2)$ -torus link, and that any $(n3, n2)$ -torus link must be a collection of n of these vertices. In

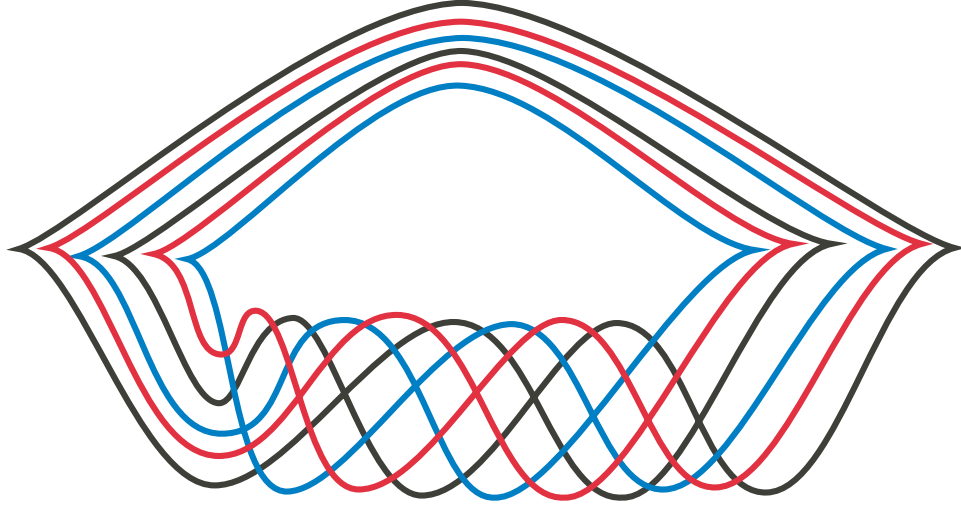


Figure 3.2: $3(3, 2)$ Torus Link with all components having maximal Thurston-Bennequin invariant.

order to prove this theorem, we will follow the strategy outlined by Etnyre and Honda in [9]. That is, we will first classify all links L with maximal Thurston-Bennequin invariant in all components. In the case of positive (np, nq) -torus links, we will show that there is a unique such L up to Legendrian isotopy. Second, we show that if any component of L does not have maximal Thurston-Bennequin invariant then it is possible to destabilize that component. Repeating this, we see that the link destabilizes to the unique one with maximal Thurston-Bennequin invariant in all components. Thus, our proof of the above theorem will be complete with the proof of the following three lemmas.

Lemma 3.0.7. *There exists a (np, nq) -torus link with components K_1, \dots, K_n such that each component has maximal Thurston-Bennequin invariant, i.e.*

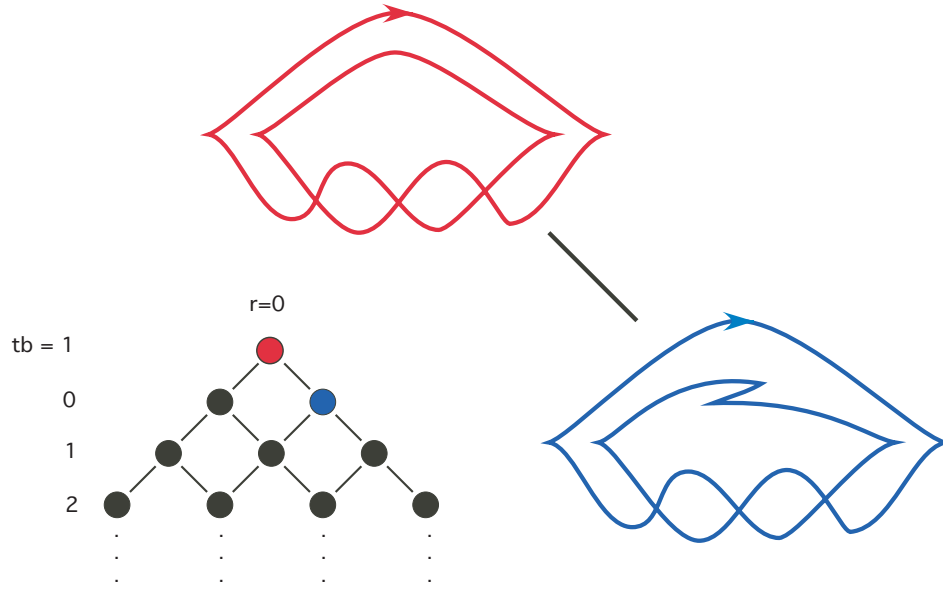


Figure 3.3: All $(3, 2)$ Torus knots are stabilizations of the unique one with maximal Thurston-Bennequin invariant.

$$tb(K_i) = pq - p - q \text{ for } i = 1, \dots, n.$$

Proof. An (np, nq) -torus link with maximal Thurston-Bennequin invariant can be explicitly constructed; for example Figure 3.2 shows a $3(3, 2) = (9, 6)$ -torus link with each component having maximal Thurston-Bennequin invariant. The following alternate construction will be useful for when we later want to see uniqueness. Let T be a convex torus bounding an unknotted solid torus with two dividing curves of slope -1 and ruling slope $\frac{q}{p}$. Let L be n ruling curves on T . Clearly L is an (np, nq) -torus link. Each component of L intersects the dividing curves $2 \left| \det \begin{pmatrix} p & -1 \\ q & 1 \end{pmatrix} \right| = 2(p + q)$ times. Thus by Theorem 2.2, $tw(K_i, T)$, the twisting of the contact planes along a compo-

ment K_i of L measured with respect to T , is $-p - q$. Since the framing on the normal bundle of a (p, q) -torus knot given by its Seifert surface differs from the one induced by T by pq , we have $tb(K_i) - tw(K_i, T) = pq$. (See Remark 2.2.4.) It then follows that each component K_i of L has $tb(K_i) = pq - p - q$, and thus L has maximal Thurston-Bennequin invariant. \square

Lemma 3.0.8. *Let L and L' be two topologically isotopic Legendrian positive torus links with each component having maximal Thurston-Bennequin invariant. Then L and L' are Legendrian isotopic.*

Proof. Let L and L' be two topologically isotopic Legendrian positive torus links with each component having maximal Thurston-Bennequin invariant. Let T and T' be the two tori on which L and L' sit. By the Relative Convex Realization Principle, Theorem 2.2, we may make T convex without moving L since the twisting of each component K with respect to T is $-p - q$ which is negative. Since T is convex and the Thurston-Bennequin invariant is maximal for each component, we may assume that T is in standard form with the components of L as a subset of the ruling curves of T . If not, then $\#(\Gamma \cap K) > |\Gamma \cap K|$ where $\#(\Gamma \cap K)$ is the unsigned intersection number and $|\Gamma \cap K|$ is the geometric intersection number. This implies the existence of a bypass and hence a destabilization of K . Since all components K are assumed to have maximal Thurston-Bennequin invariant, K cannot be destabilized. Hence, T must be in standard form. Let $\frac{-r}{s}$, $r, s > 0$, be the slope of the dividing curves Γ and $2n$ the number of dividing curves. According to Lemma 2.2.2,

for each component K of L ,

$$\begin{aligned}
 tb(K) &= pq - \frac{1}{2} \#(\Gamma \cap K) \\
 &= pq - n \left| \det \begin{pmatrix} p & -s \\ q & r \end{pmatrix} \right| \\
 &= pq - n(pr + sq) \\
 &= pq - npr - nsq
 \end{aligned}$$

So for $tb(K) = pq - p - q$, we must have that $n = 1$ and $r = s = 1$. Therefore, the slope of the dividing curves is -1 and the number of dividing curves is 2. Up to contact isotopy, there is a unique convex torus in S^3 which bounds an unknotted solid torus, has dividing slope -1 and two dividing curves. (See Lemma 2.2.4 for details.) Repeating the above argument for L' and T' , we see that L and L' sit as ruling curves on the same convex torus T , and hence are isotopic. \square

Lemma 3.0.9. *Let L be an oriented Legendrian (np, nq) -torus link with $p, q > 0$. Then for each component K_i of L with $tb(K_i) = pq - p - q - n_i$, there exist positive integers n_{i1} and n_{i2} such that $n_i = n_{i1} + n_{i2}$, $r(K_i) = n_{i2} - n_{i1}$, and $K_i = S_-^{n_{i1}}(S_+^{n_{i2}}(K'))$, where K' is the unique Legendrian (p, q) -torus knot with maximal Thurston-Bennequin invariant.*

Proof. As in the proof of Lemma 3.08 we can assume L sits on a standardly embedded convex torus T . If T is not in standard form, then we can destabilize by following the steps in the proof of Lemma 3.08. Therefore, assume the

slope $\frac{-r}{s}$ of the dividing curves is not equal to -1 or the number of dividing curves is not equal to 2. A first step in showing that components without maximal Thurston-Bennequin invariant destabilize is to find a torus parallel to T with exactly 2 dividing curves of slope -1 .

Case 1 $\frac{-r}{s} \neq -1$

Let $V_0 \cap V_1$ be the Heegard splitting with respect to T . As the boundary of either V_0 or V_1 , the slope of the dividing curves of T will be less than -1 . Without loss of generality, assume V_0 has this property. By Lemma 2.2.5, if we look at concentric tori in V_0 , we will see dividing curves with all slope in $[\frac{-r}{s}, 0]$. In particular, there will be a torus $T' \subset V_0$ with two dividing curves of slope -1 .

Case 2 $\frac{-r}{s} = -1$ and $n > 1$.

In an invariant neighborhood of the boundary, move T inside V_1 , then perturb T so that the ruling curves are meridional (i.e. slope = 0) and look at a convex meridional disk D . We may use the bypasses on D to reduce the number of dividing curves on the copy of T until there are only 2. Call the resulting torus T' .

Let $U = T \times [0, 1]$ be the region between T and T' in V_0 . Let A be an annulus lying between $T = T \times 0$ and $T' = T \times 1$ in U . The boundary of U is convex because both T and T' are convex, and we may assume that the ruling curves on both boundary components have slope $\frac{q}{p}$. Thus we may assume that the annulus A , having one boundary component a ruling curve K on T' and the other a component K_i of L , is convex and that its boundary is Legendrian.

We can further assume A is disjoint from the other components of L . The dividing curves will intersect K , $N = 2n \left| \det \begin{pmatrix} p & -s \\ q & r \end{pmatrix} \right|$ times, and K' , $N' = 2(p + q)$ times. As r and s are not both 1, $N' < N$ so we can find a boundary parallel arc along K among the dividing curves of A . This implies the existence of a bypass for K and hence a destabilization of L . Specifically, $K_i = S_{\pm}(K'')$ for some Legendrian K'' . Repeating this argument, we will eventually find a sequence of destabilizations so that $K_i = S_{-}^{n_{i1}}(S_{+}^{n_{i2}}(K'))$ where K' is the unique Legendrian (p, q) -torus knot with maximal Thurston-Bennequin invariant. \square

Theorem 3.0.12. (Ordered Positive Torus Link Classification) *It is possible to arbitrarily permute the components of a Legendrian positive (np, nq) -torus link.*

Proof. We prove the lemma when L is a Legendrian (np, nq) -torus link where each component has maximal Thurston-Bennequin invariant. The general case follows from Lemma 3.0.9. As in the proof of Lemma 3.0.8 we know L sits on a convex torus T as ruling curves. Since T is convex, there is a neighborhood $N = T^2 \times [-1, 1]$ of T such that $T^2 \times \{0\} = T$ and the contact structure is invariant in the $[-1, 1]$ direction. So each $T^2 \times \{pt\}$ is foliated by ruling curves of slope $\frac{q}{p}$. We can isotop each component of L to a different torus in N then further isotop the components on the different levels so that their order is permuted by any preassigned permutation. Finally we isotop the permuted components back to T . \square

It is also possible to show that arbitrary permutations of an (np, nq) -torus link directly using Legendrian Reidemeister moves. On the following pages, the general procedure is shown through the example of the $3(3, 2) = (9, 6)$ link with maximal Thurston-Bennequin invariant in all components. Recall that in order to show that all permutations are possible, it suffices to show that all permutations are possible for the 3-copy. A similar sequence of moves can be used for any (np, nq) -torus link simply by repeating the steps shown here however many times is necessary.

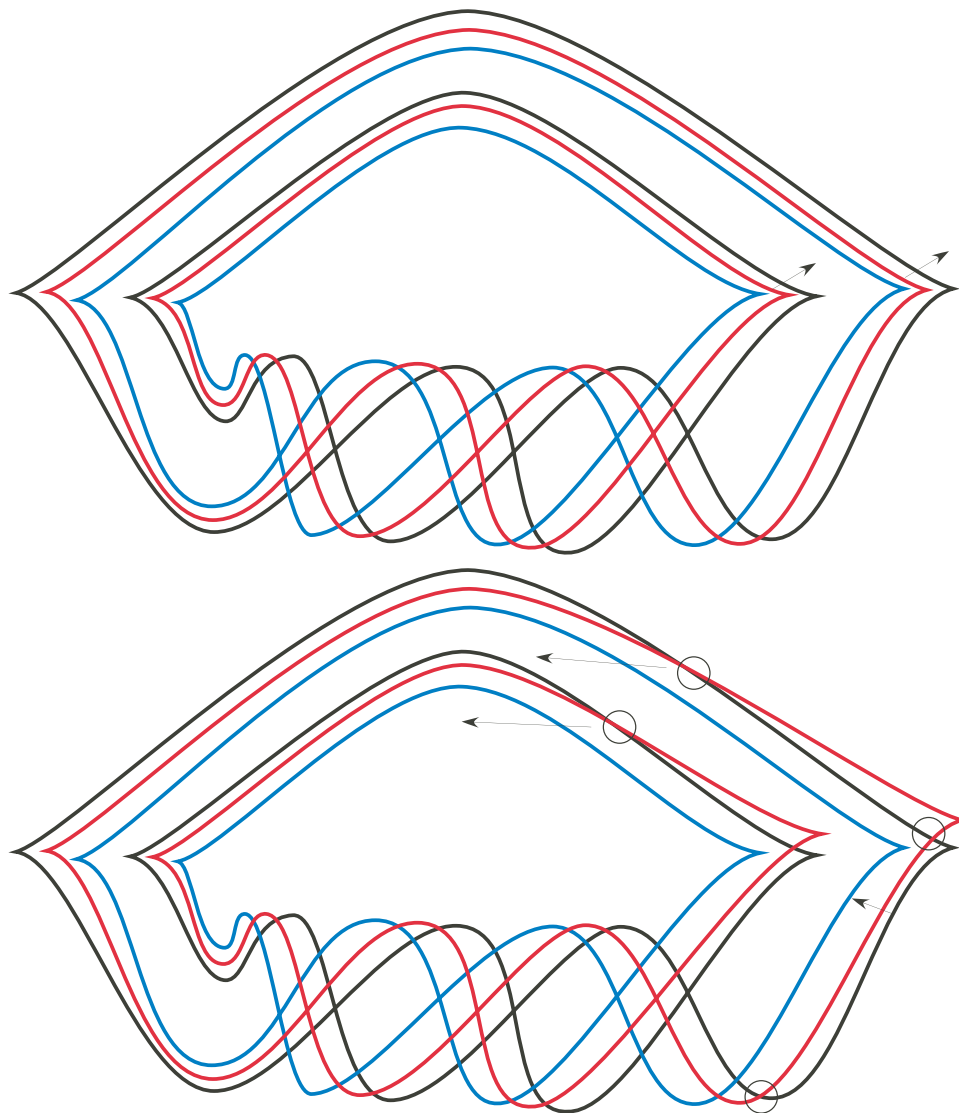


Figure 3.4: *Top:* To show that the black and red components can be permuted while leaving the blue component fixed, our first step is to push the right red cusps through the black component as indicated by the arrows. *Bottom:* The next step is to push the rightmost black cusp back through the red as shown, and to slide the two black over red crossings over to the left.

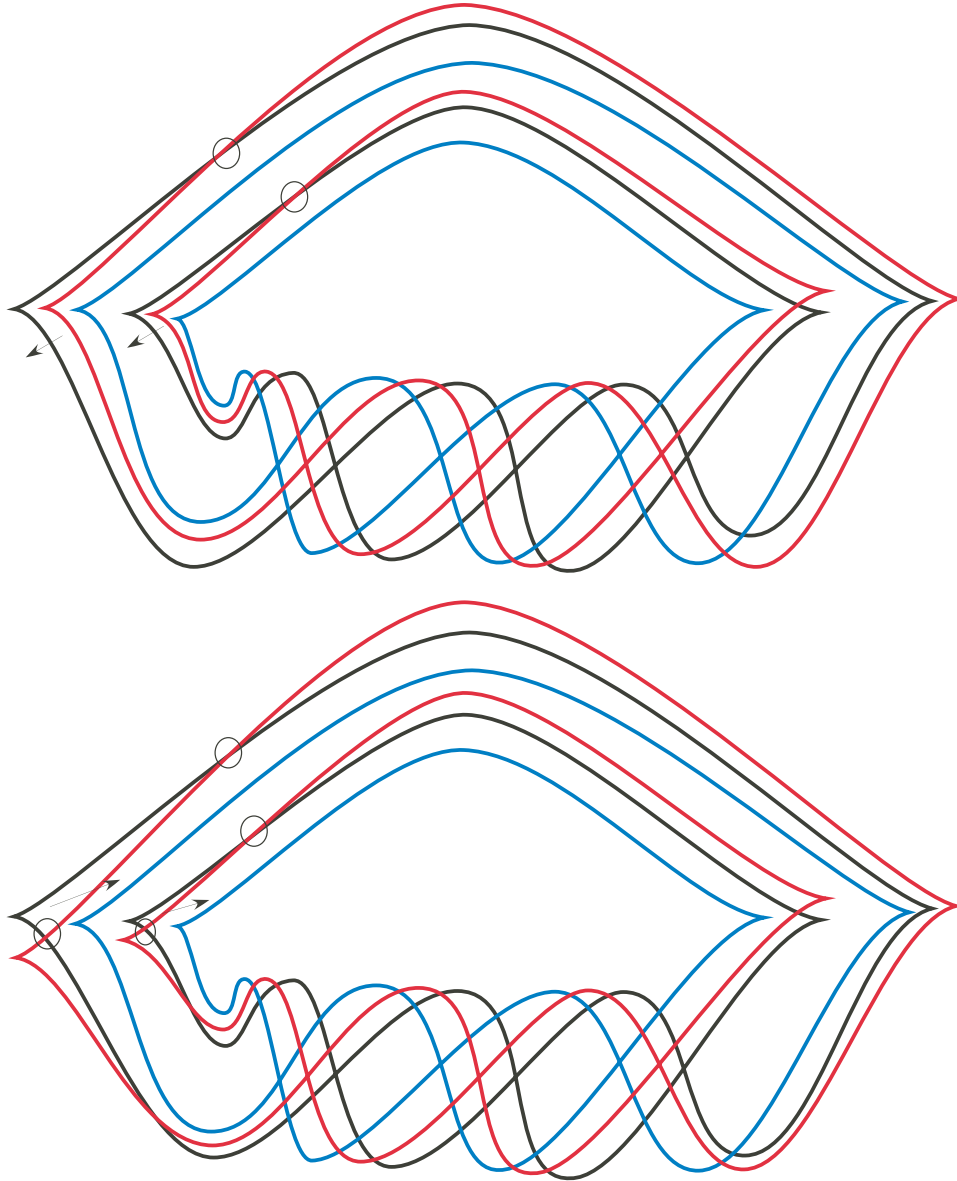


Figure 3.5: *Top*: Next push the red cusps on the left hand side through the black component as indicated by the arrows. *Bottom*: Now push the left black cusps back through the red as shown.

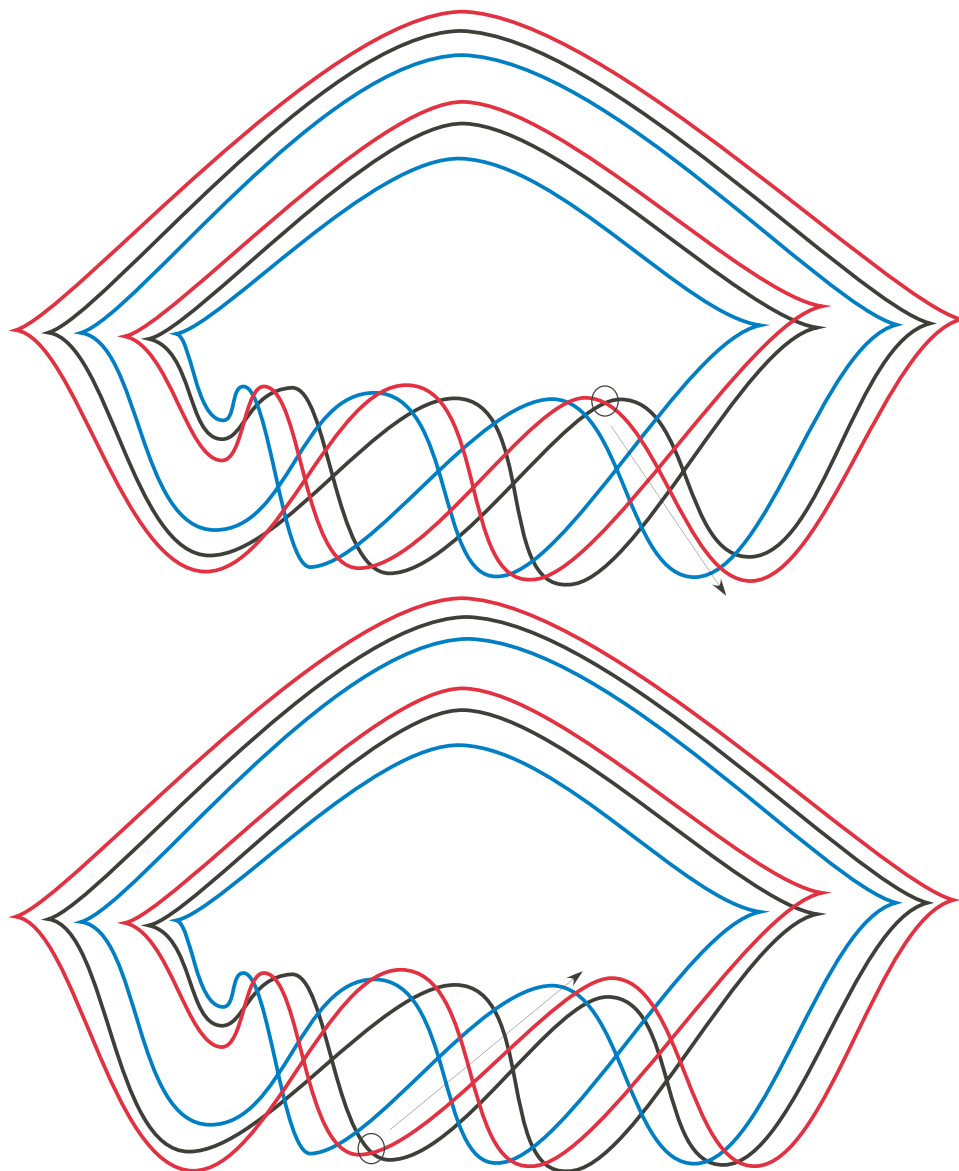


Figure 3.6: *Top:* Use repeated triple point moves to slide the indicated crossing from the top to the bottom. *Bottom:* Use another series of triple point moves to move the circled crossing from bottom to top.

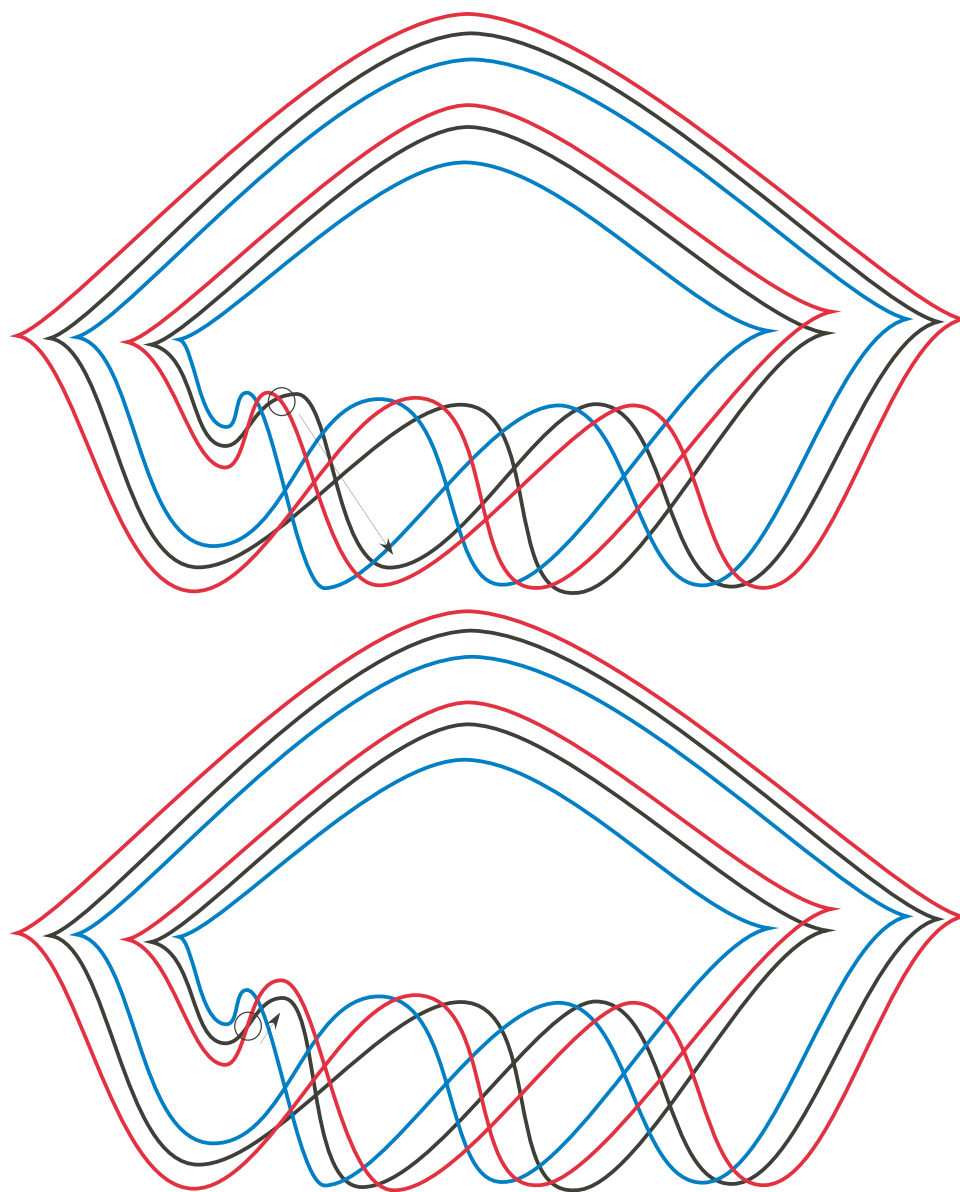


Figure 3.7: In both figures above, use triple point moves to slide the circled crossings to the indicated locations.

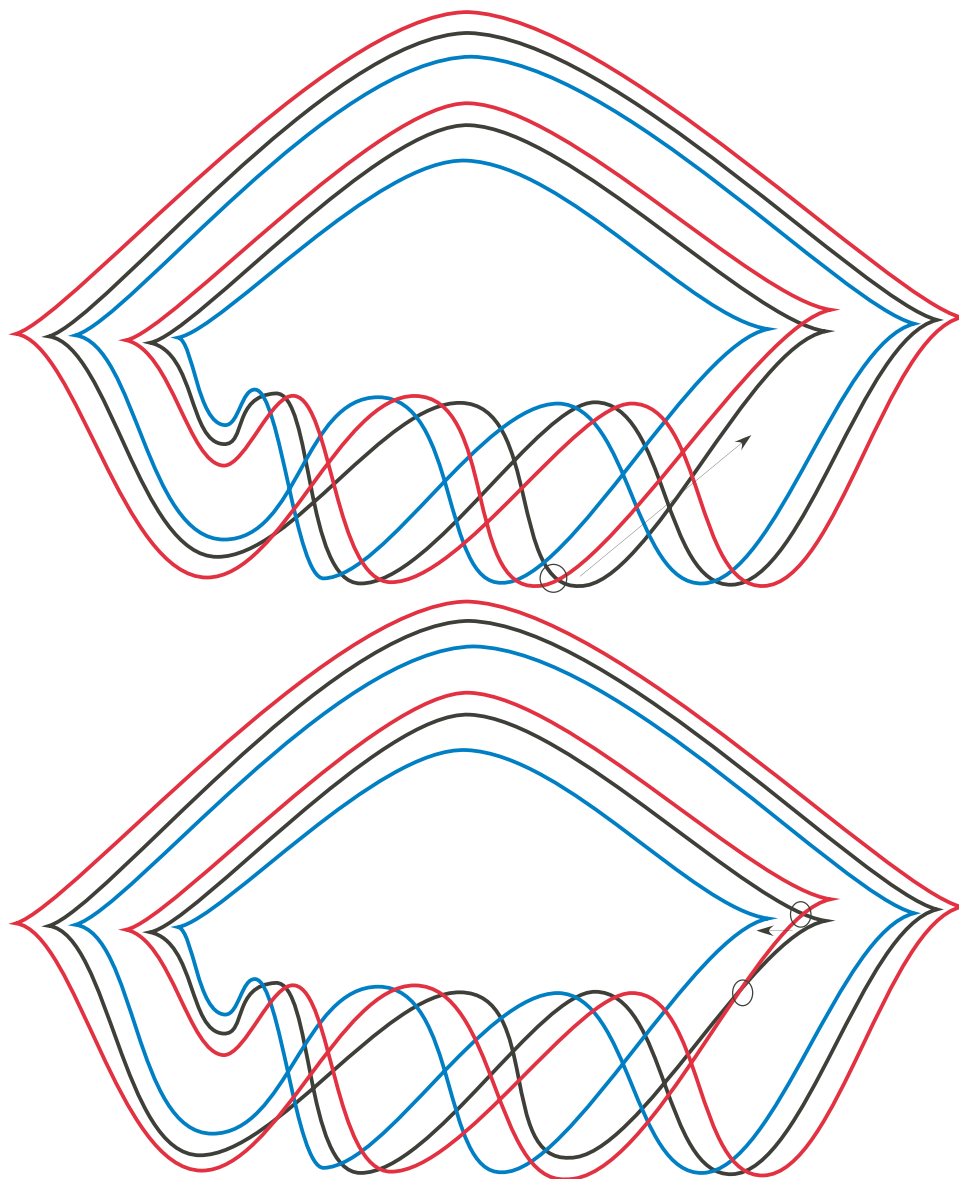


Figure 3.8: *Top:* Use repeated triple point moves to slide the indicated crossing from the bottom to the top. *Bottom:* Now push the inner black cusp on the right hand side back through the red strand as shown.

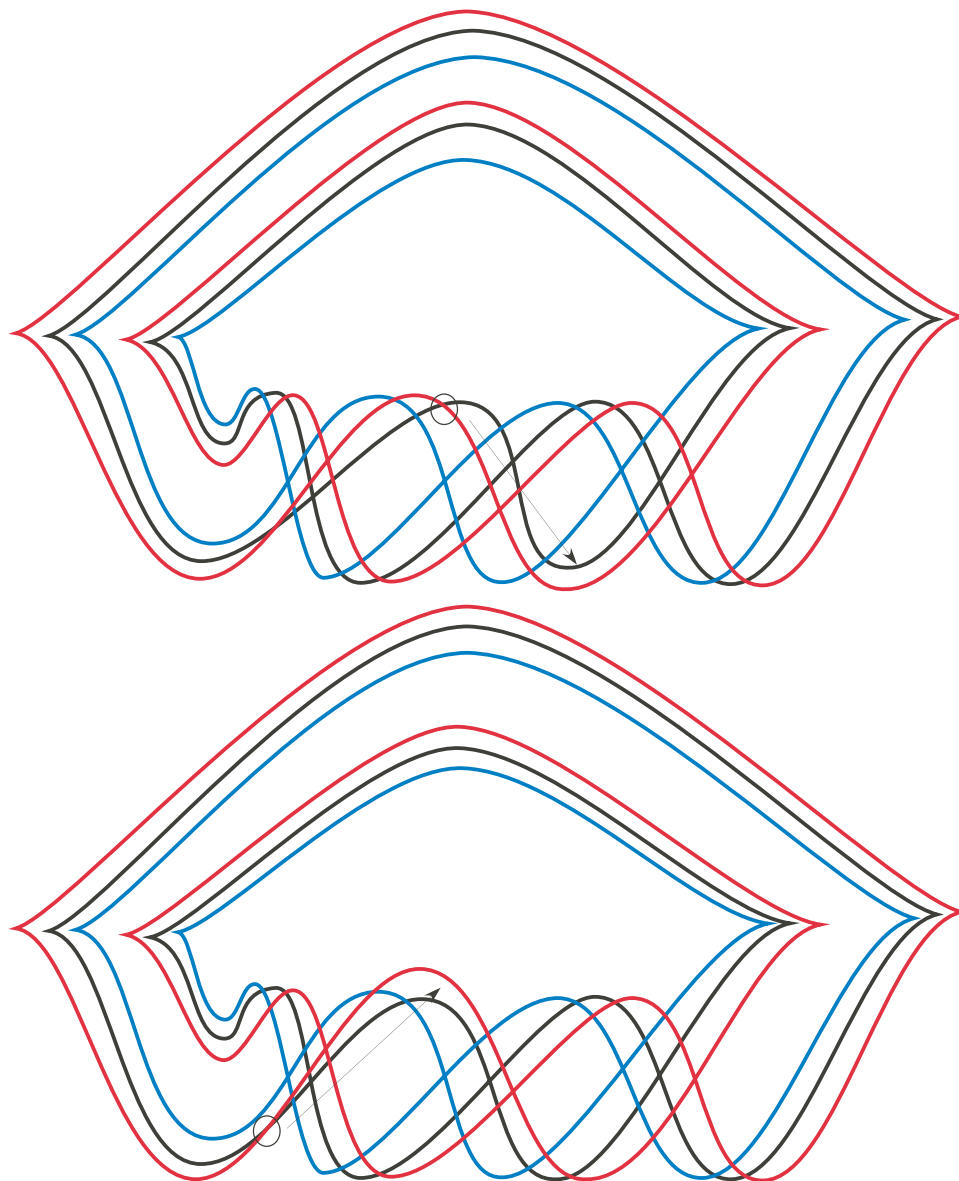


Figure 3.9: In both figures above, use triple point moves to slide the circled crossings to the indicated locations. After these two sets of moves, the red and black component will be successfully permuted.

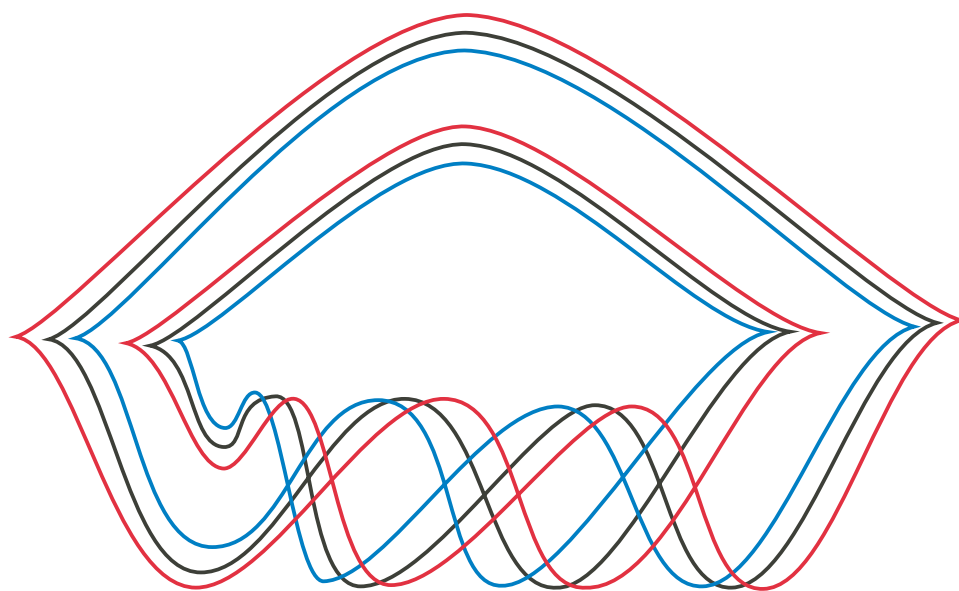


Figure 3.10: We have now shown that it is possible to permute the red and black strands while leaving the blue fixed. Hence all permutations of the n -copy of $(3, 2)$ are possible.

Chapter 4

Negative Torus Links

Let K be a negative (p, q) -torus knot with maximal $tb = pq$, p and q relatively prime, $p < 0$, and $|p| > q > 1$. By specifying $q > 1$, we are only considering nontrivial negative torus knots. We have that $|p| = mq + e$ for $q > e > 0$. Then the different possibilities for K are determined by the possible values of n_1 and n_2 where $m = n_1 + n_2 + 1$, $n_1, n_2 \geq 0$. The value of n_1 is the number of twists on the left hand side of the knot, and the value of n_2 is the number of twists on the right hand side of the knot, as pictured in Figure 4.1. Note that the knot will have $L = (n_1 + 1)q$ left cusps on the left hand side of the diagram, and $R = (n_2 + 2)q + e$ right cusps on the right hand side of the diagram. For example, the knot $(-5, 2)$ has two possible unoriented versions since $5 = 2(2) + 1$, and so $2 = n_1 + n_2 + 1$. Therefore, we have either that $n_1 = 0$ and $n_2 = 1$ or that $n_1 = 1$ and $n_2 = 0$. In the first case, given an orientation, the rotation number of the knot is equal to ± 3 , $L = 2$, and

$R = 5$. In the second case, the rotation number of the knot is equal to ± 1 , $L = 4$, and $R = 3$. The two knots are shown in Figure 4.2. When $\frac{p}{2} < q < p$ then $m = 1$ and $n_1 = n_2 = 0$ and so there will be only one unoriented version of the knot. The diagrams of $(-4, 3)$ and $(-5, 4)$ are shown in Figures 5.7 and 4.3 respectively. All other negative (p, q) -torus knots are stabilizations of the ones with maximal Thurston-Bennequin invariant. This relationship can be shown by a mountain range diagram like the one in Figure 4.5 for $(-5, 2)$ knots and in Figure 4.4 for $(-7, 3)$ knots.

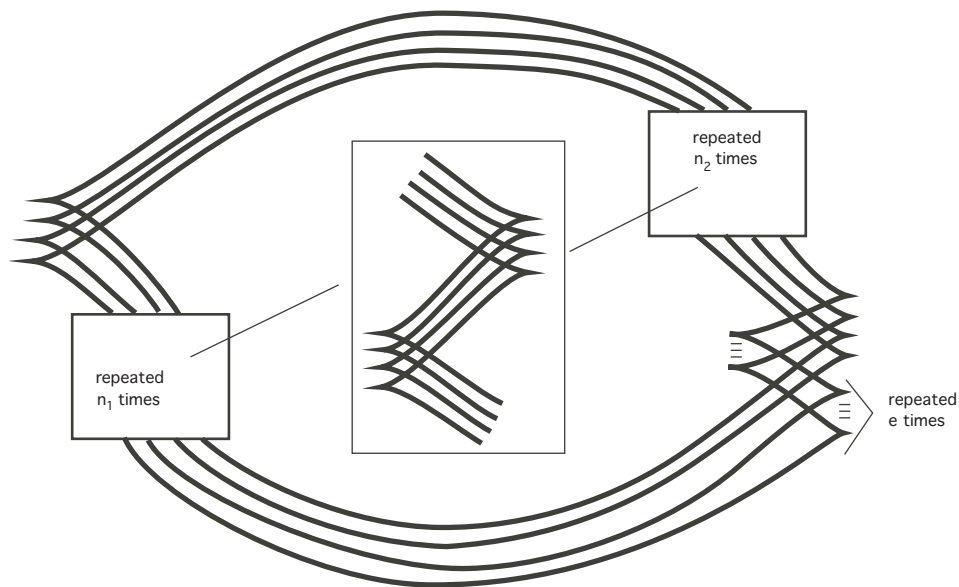


Figure 4.1: The front projection of a Legendrian negative (p, q) -torus knot.

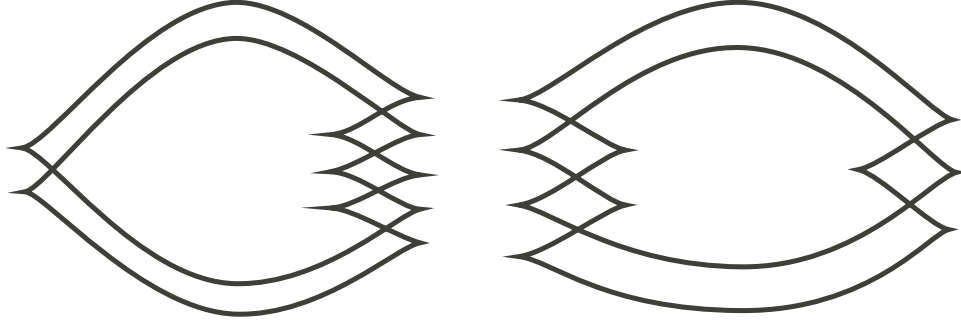


Figure 4.2: Two versions of $(-5, 2)$ with maximal $tb = -10$. The version on the left has $r = \pm 3$, and the one on the right has $r = \pm 1$.

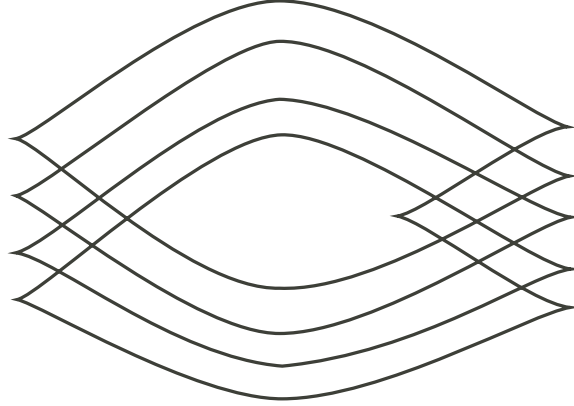


Figure 4.3: $(-5, 4)$ with maximal $tb = -20$.

Theorem 4.0.13. (Unordered Negative Torus Link Classification) *Let L be an unordered Legendrian (np, nq) -torus link with components K_1, \dots, K_n , and L' be an unordered Legendrian (np, nq) -torus link with components K'_1, \dots, K'_n . Then L and L' are Legendrian isotopic if and only if there exists a permutation σ of $\{1, 2, \dots, n\}$ such that $tb(K_i) = tb(K'_{\sigma(i)})$ and $r(K_i) = r(K'_{\sigma(i)})$.*

In order to prove this theorem, we will follow the strategy outlined by Etnyre and Honda in [9]. This is similar to the strategy followed for positive

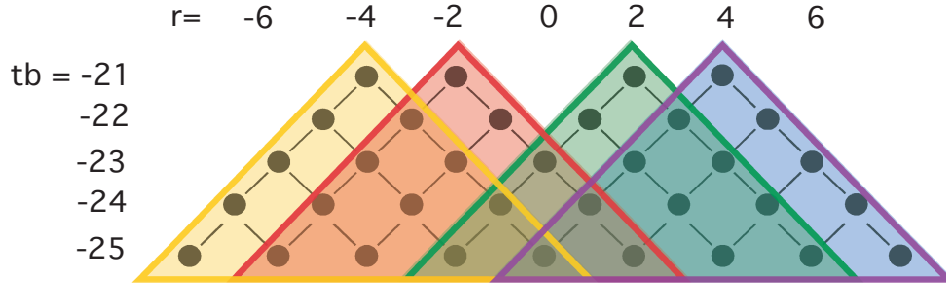


Figure 4.4: This mountain range diagram shows all possible $(-7, 3)$ -torus links are stabilizations of the ones with maximal Thurston-Bennequin invariant. Notice that if a vertex is in only one colored triangle, then it can only destabilize to a unique vertex at the top of the mountain range. If a vertex is in more than one colored triangle, then it is always possible to destabilize to two vertices at the top of the mountain range that differ by $2e$ or $2(q - e)$ (or perhaps both).

torus links in the previous chapter, but it is more complicated due to the more complex form of negative torus knots. We will first classify all links L with maximal Thurston-Bennequin invariant in all components. In the case of negative (np, nq) -torus links, we will show that there is a unique link up to Legendrian isotopy for each of the $2m$ possible rotation numbers of a negative (p, q) torus knot. Second, we show that if any component of L does not have maximal Thurston-Bennequin invariant then that component destabilizes, and so the link destabilizes to one with maximal Thurston-Bennequin invariant in all components. Last, we show the relationship between certain stabilizations of different links with maximal Thurston-Bennequin invariant in each component.

Lemma 4.0.10. *Let L be a Legendrian negative (np, nq) -torus link with components K_1, \dots, K_n where $n > 0$, $p < 0$, and $q > 1$. If $tb(K_1) + \dots + tb(K_n) <$*

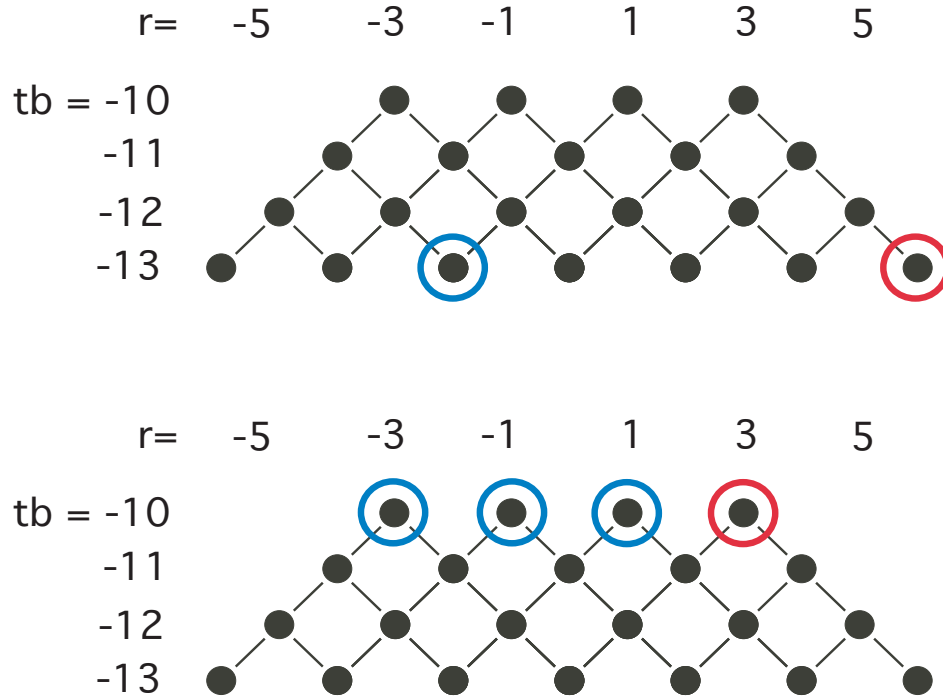


Figure 4.5: This mountain range diagram shows all possible $(-5, 2)$ -torus links are stabilizations of the ones with maximal Thurston-Bennequin invariant. Further, the two versions of $(-5, 2)$ circled in the top figure cannot be two components of the same link as they have no possible destabilizations in common as shown in the bottom figure.

npq then there exists a Legendrian (np, nq) -torus link L' with components K'_1, \dots, K'_n such that $tb(K'_1) + \dots + tb(K'_n) > tb(K_1) + \dots + tb(K_n)$ and L is a stabilization of L' .

Proof. Let K be a component of L with $tb(K) < pq$. Then by Remark 2.2.4 $tw(K) - pq < 0$, and so by Lemma 2.2.1 we can assume that K lies on a standardly embedded convex torus T . We know that the dividing curves Γ on T have slope $\frac{-r}{s} \neq \frac{q}{p}$ since $tb(K) = pq - \frac{1}{2}\#(\Gamma \cap K) < pq$. Let $V_0 \cup V_1$ be

the Heegard splitting of S^3 associated to T . Now as measured on either V_0 or V_1 , the dividing curves have slope less than $\frac{q}{p}$. Assume V_0 has this property. By Lemma 2.2.5, we can find tori in V_0 whose dividing curves have any slope in $[\frac{-r}{s}, 0)$. In particular, we can find a torus T' in V_0 whose dividing curves have slope $\frac{q}{p}$. As in the proof of Lemma 3.0.9, we can take an annulus A between T and T' with one boundary component on K and the other on a Legendrian divide of T' (and disjoint from the other components of L). The dividing curves will intersect K , $N = \left| \det \begin{pmatrix} p & -s \\ q & r \end{pmatrix} \right| = pr + sq > 0$ times, and the Legendrian divide $N' = \left| \det \begin{pmatrix} p & p \\ q & q \end{pmatrix} \right| = 0$ times. Thus, $N' < N$ so we can find a boundary parallel arc along K among the dividing curves of A . By the Imbalance Principle, Proposition 2.2.1, this implies the existence of a bypass for K and hence a destabilization of L . \square

Lemma 4.0.11. *Let L be a Legendrian negative (np, nq) -torus link with each component of L having maximal Thurston-Bennequin invariant. Then each component of L must have the same rotation number. In particular, this means that L is an n -copy of a (p, q) -torus knot with maximal Thurston-Bennequin invariant.*

Proof. Let K be any component of L . Let T be a standardly embedded torus on which K sits. By Theorem 2.2, we may isotop T so that it is convex since $pq = tb(K) = pq - \frac{1}{2}\#(\Gamma \cap K)$, and so $\frac{1}{2}\#(\Gamma \cap K) = 0$, i.e. the twist is zero. By Giroux Flexibility, we can put T_ξ in standard form. Since $\frac{1}{2}\#(\Gamma \cap K) = 0$,

K is a Legendrian divide. Similarly, all components of L are Legendrian divides. Since all of the Legendrian divides of T are Legendrian isotopic, all components must have the same rotation number. \square

Note that the above results say that we can only make an (np, nq) -torus link using n components that have a path down from the same component with maximal Thurston-Bennequin invariant at the top of the mountain range diagram. Otherwise, we will have components that destabilize to components with maximal Thurston-Bennequin invariant and different rotation numbers, which the above claim shows is impossible. For example, the two $(-5, 2)$ knots shown in Figure 4.5 with rotation numbers -2 and 6 cannot be linked together to form a $(-10, 4)$ -torus link as they do not have a common destabilization with maximal Thurston-Bennequin invariant. Another way to see this is to use colored triangles like the ones in Figure 4.4 for $(-7, 3)$ -torus links. Only vertices that lie in the same colored triangle can be linked together. The vertex at the center of the bottom row with $tb = -25$ and $r = 0$ can be linked with any other vertex in the diagram since it is in all four colored triangles, i.e. it can be realized as a stabilization of all of the vertices on the top row. On the other hand, the two vertices at the edges of the bottom row cannot be linked together as they each have only one destabilization, and those destabilizations are not the same.

Remark 4.0.14. The components of L as knots may have more than one possible destabilization, but as components of L the destabilizations are more

restricted. \diamond

Lemma 4.0.12. *Let L and L' be two topologically isotopic unordered Legendrian negative torus links with each component of L and L' having maximal Thurston-Bennequin invariant. Then L and L' are Legendrian isotopic if and only if L and L' are n -copies of the same (p, q) -torus knot.*

Proof. As shown in Lemma 4.0.11, all of the components of L are Legendrian isotopic and thus have the same Thurston-Bennequin and rotation number invariants. Similarly, all of the components of L' have the same Thurston-Bennequin and rotation number invariants. As stated in the introduction, a clearly necessary condition for the equivalence of two Legendrian torus links L and L' is that we be able to pair up each component of L with one from L' so that the elements of each pair have the same Thurston-Bennequin invariant and rotation number. We know by the proof of Theorem 1.0.2 in [9] that two Legendrian negative torus knots K and K' with maximal Thurston-Bennequin invariant are Legendrian isotopic if and only if $r(K) = r(K')$, it must be the case that the rotation number of the components of L agree with the rotation number of the components of L' . Hence, L Legendrian isotopic to L' implies that L and L' are n -copies of the same (p, q) -torus knot.

Let L be an (np, nq) -torus link with components (K_1, \dots, K_n) such that $tb(K_i) = pq$, for each i . Let T be a torus bounding an unknotted solid torus on which L sits. Since $tw(K_i, T) = 0$ we may isotop T , relative to L , so that it is convex. Moreover, since $tw(K_i, T) = 0$, each component K_i must be disjoint from the dividing curves. We may thus take L to be

a subset of the Legendrian divides when T is isotoped to be a convex torus with characteristic foliation in standard form. Similarly, we can take L' to be a subset of the dividing curves on T' , where T' is a torus bounding an unknotted solid torus on which L' sits. It remains to show there is a contact isotopy taking T to T' . By Theorem 2.2.8, the contactomorphism type of the solid torus bounding T (resp. T') is determined by the number of positive regions on a convex meridional disk with Legendrian boundary. By Lemma 2.2.2, the the number of positive regions on a convex meridional disk with Legendrian boundary is determined by the rotation number of the components of L (resp. L'). Since L and L' are n -copies of the same (p, q) -torus knot, this number is the same for both T and T' . Hence, we can apply Theorem 2.2.8 and Theorem 2.1.2 to find a contact isotopy taking T to T' . Thus, L and L' sit as Legendrian divides on the same convex torus, and so L and L' are Legendrian isotopic. \square

In the following lemma, $S_{-,all}^m(L)$ means that all components of L have been negatively stabilized m times. Similarly, $S_{+,all}^m(L)$ denotes m positive stabilizations of all n components. We will sometimes refer to such stabilizations of all n components as n -fold stabilizations of L .

Lemma 4.0.13. *Let L and L' be two topologically isotopic Legendrian negative torus links with each component having maximal Thurston-Bennequin invariant. If the rotation numbers of each component of L and L' are r and $r - 2e$, respectively, then $S_{-,all}^e(L)$ and $S_{+,all}^e(L')$ are Legendrian isotopic. If*

the rotation numbers of each component of L and L' are r and $r - 2(q - e)$, respectively, then $S_{-,all}^{q-e}(L)$ and $S_{+,all}^{q-e}(L')$ are Legendrian isotopic.

Proof. As in the proof of Lemma 4.0.12 above, we can take L and L' to lie as subsets of the Legendrian divides of tori T and T' . We know T and T' are contained in basic slices: if $|p| = mq + e$, $0 < e < q$, by Lemma 2.2.5 there are solid tori V_m, V'_m containing T, T' , respectively, with boundaries T_m, T'_m that are standard neighborhoods of unknots with $tb = -m$. The rotation numbers of these unknots determine the rotation numbers of the components of L and L' , as explained in Chapter 2. The facts outlined in Chapter 2 concerning the f_T invariant are used below. First consider the case where the rotation numbers of the components of L and L' differ by $2e$. We can deduce that V_m and V'_m are standard neighborhoods of an unknot with the same rotation number as follows. If $f_T(\mu) = 1 - q$ then,

$$\begin{aligned}
 r(K_i) &= pf_T(\mu) + qf_T(\lambda) \\
 &= p(1 - q) + qf_T(\lambda) \\
 &= p - pq + q(f_m(\lambda) + m - |p|) \\
 &= p - pq + qf_m(\lambda) + mq - |p|q \\
 &= qf_m(\lambda) - |p| + mq \\
 &= qf_m(\lambda) - e
 \end{aligned}$$

Similarly, we get that if $f_{T'}(\mu) = 1 - q$, then

$$r(K'_i) = qf'_m(\lambda) + e.$$

In this case, we have that $r(K'_i) - r(K_i) = 2e$ if and only if $f_m(\lambda) = f'_m(\lambda)$. Thus V_m and V'_m are standard neighborhoods of an unknot with the same rotation number. In particular, we can assume $T_m = T'_m$; let T'' denote this torus. Choose the ruling slope on T'' to be $\frac{q}{p}$, and let L'' be any collection of n ruling curves on T'' . By connecting L to L'' with n annuli one easily sees that L'' is $S^e_{-,n}(L)$ since the dividing curves do not intersect the boundary component on T and intersect the other boundary component,

$$N = 2 \left| \det \begin{pmatrix} p & -m \\ q & 1 \end{pmatrix} \right| = 2(p + qm) = -2e \text{ times.}$$

Thus $S^e_{-,n}(L)$ sits as n ruling curves on T'' . One can similarly realize $S^e_{+,n}(L')$ as n ruling curves on T'' . Since the ruling curves are isotopic, this gives us that $S^e_{-,n}(L)$ and $S^e_{+,n}(L')$ are isotopic. If the rotation numbers of the components of L and L' differ by $2(q - e)$, a calculation similar to the one above gives that $r(K) = q(f_{m+1}(\lambda) + 1) - e$ and $r(K'_i) = q(f'_{m+1}(\lambda) + 1) + e$, and so $f_{m+1}(\lambda) = f'_{m+1}(\lambda)$. Thus V_{m+1} and V'_{m+1} are standard neighborhoods of an unknot with the same rotation number. The proof is finished by a similar argument to the one above. \square

Before giving the proof of the theorem below, there are some subtleties in the similar proof for knots in [9] that are worth explaining here. Suppose K and K' are two negative (p, q) -torus knots with the same invariants that destabilize to K_d and K'_d and the rotation numbers of K_d and K'_d differ by $2i$. The argument in [9] carefully explains why K and K' are isotopic if $i = e$ or $i = q - e$. To see why K and K' are isotopic for any value of i ,

we will look at an example. Consider two $(-7, 3)$ torus knots, K and K' , with $tb = -25$ and $r = 0$. In Figure 4.4, these knots are both represented by the center dot on the bottom line of the figure. Suppose K destabilizes to K_d with $tb = -21$ and $r = 4$, and K' destabilizes to K'_d with $tb = -21$ and $r = -2$. Note that in this case $2e = 2$ and $2(q - e) = 4$, while the rotation numbers of K_d and K'_d differ by 6. We have that $K = S_-^4(K_d) = S_-^3(S_-(K_d))$. Since $e = 1$, we have that $S_-(K_d) = S_+(K''_d)$ for K''_d with $tb = -21$ and $r = 2$. So $K = S_-^3(S_-(K_d)) = S_-^3(S_+(K''_d)) = S_+(S_-^3(K''_d))$. We also have that $K' = S_-(S_+^3(K'_d)) = S_-(S_+(S_+^2(K'_d)))$. Now $S_+^2(K'_d) = S_-^2(K''_d)$ since their rotation numbers differ by $2(q - e) = 4$. Therefore, $S_-(S_+(S_+^2(K'_d))) = S_-(S_+(S_-^2(K''_d))) = S_+(S_-^3(K''_d))$. We have already shown that $K = S_+(S_-^3(K''_d))$, hence K and K' are isotopic. In general, if the rotation numbers of K_d and K'_d differ by a value greater than $2e$ or $2(q - e)$ then there are other destabilizations of K and/or K' whose rotation numbers will differ by $2e$ or $2(q - e)$ and the result will follow.

Proof of Theorem. Let L and L' be two (np, nq) -torus links with the same invariants, that is that there is a pairing of components of L and L' so that the Thurston-Bennequin and rotation invariants are the same within each pair. First look at all of the possible destabilizations of each component of L and L' . If any component K of L has only one possible destabilization, say K_d , then the entire link must destabilize to the n -copy of K_d . This follows from the fact that we have shown that a link that has maximal Thurston-Bennequin in all components also has the same rotation number in each component.

Similarly for K' a component of L' . Hence both L and L' destabilize to an n -copy of a knot with maximal Thurston-Bennequin invariant. We have also shown that up to Legendrian isotopy there is a unique n -copy of a knot with maximal Thurston-Bennequin invariant. Therefore, L and L' are the same stabilizations of the same link, and so are isotopic. Note that if K is any component of L such that $tb(K) > \overline{tb} - \min(q - e, e)$, then there will only be one choice for K to destabilize to on the top row of the mountain range diagram, and so we are done unless all components have Thurston-Bennequin invariant greater than $\overline{tb} - \min(q - e, e)$.

Now assume that all components of L, L' have Thurston-Bennequin invariant $\leq \overline{tb} - \min(q - e, e)$. It suffices to show that if L and L' destabilize to L_d and L'_d and the rotation numbers of the components of L_d and L'_d differ by $2i$ then both L and L' are stabilizations of $S_{-,all}^i(L_d)$ and $S_{+,all}^i(L'_d)$, for $i = e$ or $i = q - e$. Let K (K') be a component of L (L') that has the greatest Thurston-Bennequin invariant among the components of L (L'), say $tb(K) = tb(K') = \overline{tb} - j$ for some $j \geq \min(q - e, e)$. Consider the case where K destabilizes to K_d and K' destabilizes to K'_d so that K_d and K'_d both have maximal Thurston-Bennequin invariant and $r(K'_d) = r(K_d) + 2e$. Note that in this case we must have that $j \geq e$. Since K destabilizes to K_d , we have that $K = S_-^{j_1}(S_+^{j_2}(K_d))$ for some $j_1, j_2 \geq 0$, $j_1 + j_2 = j$. Similarly, $K' = S_+^{j'_1}(S_-^{j'_2}(K'_d))$ for some $j'_1, j'_2 \geq 0$, $j'_1 + j'_2 = j$. Now $r(K) = r(K_d) + j_2 - j_1$

and $r(K') = r(K'_d) + j'_1 - j'_2$. Since $r(K) = r(K')$, we have

$$r(K_d) + j_2 - j_1 = r(K'_d) + j'_1 - j'_2 = r(K_d) + 2e + j'_1 - j'_2.$$

Thus $j_2 - j_1 = 2e + j'_1 - j'_2$. Solving $j_1 + j_2 = j = j'_1 + j'_2$ for j_2 gives $j_2 = j'_1 + j'_2 - j_1$. Substituting in for j_2 , we get $j'_1 + j'_2 - 2j_1 = 2e + j'_1 - j'_2$. Solving this equation for j'_2 gives $j'_2 = j_1 + e$. Thus $j'_2 \geq e$ since $j_1 \geq 0$. Now $j'_1 = j - j'_2 = j - j_1 - e = j_2 - e$. Thus $j_2 \geq e$ since $j'_1 \geq 0$. This gives us that

$$K = S_-^{j_1}(S_+^{j_2-e}(S_+^e(K_d)))$$

and

$$K' = S_+^{j'_1}(S_-^{j'_2-e}(S_-^e(K'_d))).$$

We know by the proof for knots in [9] that $S_+^e(K_d) = S_-^e(K'_d)$, and we have already shown that $j'_2 - e = j_1$ and $j'_1 = j_2 - e$. Therefore, K and K' are the same stabilizations of the same component. A similar argument can be applied to all of the components of L and L' since they all have Thurston-Bennequin invariant less than or equal to that of K and K' , and so less than or equal to $\overline{tb} - j$. This gives that L is a stabilization of $S_{+,all}^e(L_d)$ and L' is a stabilization of $S_{-,all}^e(L'_d)$, which are isotopic by Lemma 4.0.13 above. Further, we have shown that L and L' must be the same stabilizations of $S_{+,all}^e(L_d) = S_{-,all}^e(L'_d)$, and so L and L' are isotopic. The argument will be the same if $r(K'_d) = r(K_d) + 2(q - e)$ with $q - e$ replacing e in all of the

above formulas. If the rotation numbers of L_d and L'_d differ by something else, then an argument similar to the one for knots given before the proof of this theorem will finish the proof. \square

We show in the next chapter that not all permutations are possible for the components of a Legendrian negative (np, nq) -torus link with maximal Thurston-Bennequin invariant, i.e. $tb = pq$. We show below that all permutations are possible that preserve the cyclic ordering of the components with maximal Thurston-Bennequin invariant. In particular, all cyclic permutations of a Legendrian negative (np, nq) -torus link are possible.

Lemma 4.0.14. ([2]) *Let L be an oriented, Legendrian (np, nq) -torus link for $p < 0$ and write $L = S \cup S'$ where S is the sublinks consisting of all components with $tb = pq$. Any permutation of the components of L that preserves the cyclic ordering of the components of S can be realized by a Legendrian isotopy.*

Sketch of Proof. (See [2] for full details.) We first assume that all the components of S' have $tb = pq - 1$. We can write $S' = S_+ \cup S_-$ where the rotation number of the components of S_{\pm} are ± 1 more than the rotation number of the components of S . As in our classification arguments above we can find a convex torus T on which S sits as Legendrian divides. Now let m satisfy $|p| = mq + e$ for $0 < e < q$. We know there exist convex tori T_m and T_{m+1} that cobound $N = T^2 \times [0, 1]$, $T \subset N$ and T_m has two dividing curves of slope $-\frac{1}{m}$ and $-\frac{1}{m+1}$, respectively. The basic slice N is determined by the sign of

a bypass on a horizontal annulus in N . If the bypass is positive let T_+ be a convex torus between T_m and T with two dividing curves of slope k_+ and T_- a convex torus between T and T_{m+1} with two dividing curves of slope k_- . The slopes k_+ and k_- are chosen so that when we arrange the ruling curves of T_+ and T_- have slope $\frac{q}{p}$, the ruling curves will intersect the dividing curves twice. Choose $|S_+|$ ruling curves on T_+ and $|S_-|$ ruling curves on T_- . These ruling curves together with S make a Legendrian (np, nq) -torus link with the same invariants as L . Thus we may assume this link we just constructed is L . However, now it is clear we can arbitrarily permute the components of L in S' .

Any link with components of S' having $tb < pq - 1$ comes from a stabilization of a Legendrian link considered in the previous paragraph. Thus the non-maximal Thurston-Bennequin invariant components have the same kind of flexibility. \square

Chapter 5

A *DGA* Approach to Noncyclic Permutations

5.1 General Strategy

Let K be a negative (p, q) -torus knot with maximal tb . Let L denote the unordered 3-copy of K , and L_{123} and L_{132} denote L with different orderings. We will work with a fixed projection \bar{L} of L . The following is a description of the general strategy for proving that non-cyclic permutations are not possible for a negative (p, q) -torus link.

1. Find the *DGA* of the projection of the link. If the *DGA* has a proper augmentation, then proceed to step (3). If not, then find the *DGA* of the 2-copy of the knot which is guaranteed to have a proper augmentation by Proposition 2.3.2. In either case, let $A_{\bar{L}}$ denote this *DGA*.

2. Find all augmentations of $A_{\overline{L}}$.
3. Look at the 3-copy of either (p, q) or $(2p, 2q)$. As stated in Section 2.3, in order to show that noncyclic permutations are not possible for the n -copy of K , it suffices to show that the two 3-copies of K with strands ordered 1, 2, 3 and 1, 3, 2 are distinct.
4. For every augmentation ϵ of $A_{\overline{L}}$, look at $CH_{123}(\epsilon)$ and $CH_{132}(\epsilon)$.
5. As shown in Chapter 2, the number z_{123} (z_{132}) of distinct augmentation classes $[\epsilon]$ that yield zero divisors in $CH_{123}(\epsilon)$ ($CH_{132}(\epsilon)$) is an invariant of L_{123} (L_{132}).
 - (a) If there is a unique augmentation, then z_{123} , z_{132} are either 0 or 1. If these numbers are different, then the links are different. If these numbers are the same, go back to step (1), find the *DGA* of the 2-copy of the knot, and proceed from there.
 - (b) If there are multiple augmentations, we show that ϵ_1 and ϵ_2 represent distinct augmentation classes by showing, for example, that $CH_{123}(\epsilon_1)$, $CH_{123}(\epsilon_2)$, and $CH_{132}(\epsilon_1)$ have zero divisors, while $CH_{132}(\epsilon_2)$ does not. (See Remark 2.3.8.) We then use this to get estimates on z_{123} and z_{132} that can in turn be used to distinguish the different orderings of the link.

In Mishachev's work, the links of unknots have a unique augmentation and so z_i is 0 or 1. When $q = 2$, negative (p, q) -torus knots will have a

unique augmentation as well, and so we first attempt to distinguish these links by showing that CH_{123} has zero divisors while CH_{132} does not, or vice-versa, i.e. $z_{123} \neq z_{132}$. Unfortunately, we will show that $z_{123} = z_{132}$, and so it will be necessary to go back to step (1) and work with the double of the knot. The procedure from there will be similar to that in the section detailing the $(-4, 3)$ example. When $p = -3$, the knot does not have a proper augmentation at all. The double of a knot has at least one proper augmentation, so in order to show that noncyclic permutations of the n -copy of $(-4, 3)$ are not possible, it suffices to study $2(-4, 3)$. We begin by doing the augmentation calculations without concern about augmentation classes. Finding all of the augmentations is not a trivial task. We will show that the knot $2(-4, 3)$ has three augmentations with a minimal number of augmented vertices, call these augmentations A , B , and C , and that any other augmentation of $2(-4, 3)$ will be one of these three with additional augmented vertices. This translates to 27 possible minimal augmentations of the 3-copy of $2(-4, 3)$, call these AAA , AAB , etc. From the calculations done by hand, it appears that only the CCC augmentation has zero divisors in CH_{132} , while (at minimum) the AAA , BBB , and CCC augmentations have zero divisors in CH_{123} . When we add vertices to A , B , and C we get new augmentations $A = A_1, A_2, \dots, A_k$; $B = B_1, B_2, \dots, B_m$; and $C = C_1, C_2, \dots, C_n$ for $k, m, n \geq 1$. The calculations have been done by hand, and show that only combinations of the C_i augmentations can have zero divisors in CH_{132} . In CH_{123} , these same combinations of the C_i augmentations will

also have zero divisors, but so will the triples of any of the A_i or B_i vertices. We can then use this information to show that there are more *augmentation classes* that yield zero divisors in CH_{123} than in CH_{132} . If the conjectures supported by our calculations are true, this will give us the desired result that $z_{123} > z_{132}$ and so noncyclic permutations are not possible for the n -copy of $2(-4, 3)$. Thus noncyclic permutations are not possible for the n -copy of $(-4, 3)$.

Finally, we describe what is known about the augmentations for other negative (p, q) -torus knots and a possible strategy for showing that noncyclic permutations are not possible for the n -copy of those knots.

5.2 Set up and Labeling of Diagrams

In this section, we will describe the *DGA* associated to a projection of a negative (p, q) -torus knot. Setting up a good system of labelings is important in order to be able to see patterns in the differential and so possible augmentations of the algebra are clear.

Let K be a negative (p, q) -torus knot with maximal $tb = pq$, p and q relatively prime and $|p| > q$. We have that $|p| = mq + e$ for $q \geq e > 0$. Recall that the different possibilities for K are determined by the possible values of n_1 and n_2 where $m = n_1 + n_2 + 1$; (See Figure 4.1). Note that the knot will have $L = (n_1 + 1)q$ left cusps on the left hand side of the diagram, and $R = (n_2 + 2)q + e$ right cusps on the right hand side of the diagram.

The algebra for the entire knot projection will be easy to describe in terms of algebras associated to the left and right hand halves of the projection. The interval algebra $I_q(N)$, as defined by Mishachev [17], is the algebra associated to each side of the knot and is generated by the crossings and right cusps of the knot. $I_q(N)$ can be visualized as an interval with N marked points, and then the generators $a_{i,j}$ of $I_q(N)$ are subintervals of length $j - i$ where $1 \leq i < j \leq N$ and $j - i \leq q$. The differential of $I_q(N)$ is given by

$$\partial a_{i,j} = \begin{cases} 1 + \sum_{i < k < j} a_{i,k} a_{k,j}, & j - i = q \\ \sum_{i < k < j} a_{i,k} a_{k,j}, & j - i < q. \end{cases}$$

When $j - i = q$ the generator corresponds to a right cusp of the knot projection and when $j - i < q$ the generator corresponds to a crossing of the knot projection.

The interval algebra associated to the left hand side of a knot K is $I_q(L)$. In Figure 5.1, we see the two versions of $(-5, 2)$ with maximal Thurston-Bennequin invariant (see Figure 4.2) have been split in two in order to calculate the interval algebra. The interval algebra for the left hand side of the version of $(-5, 2)$ on the top is $I_2(2)$, and the interval algebra for the left hand side of the version of $(-5, 2)$ on the bottom is $I_2(4)$.

The interval algebra associated to the right hand side of a knot K is $I_q(R + q)$. In order to compute this, we must “borrow” cusps from the left hand side of the diagram. Given the characterization of all (p, q) torus knots

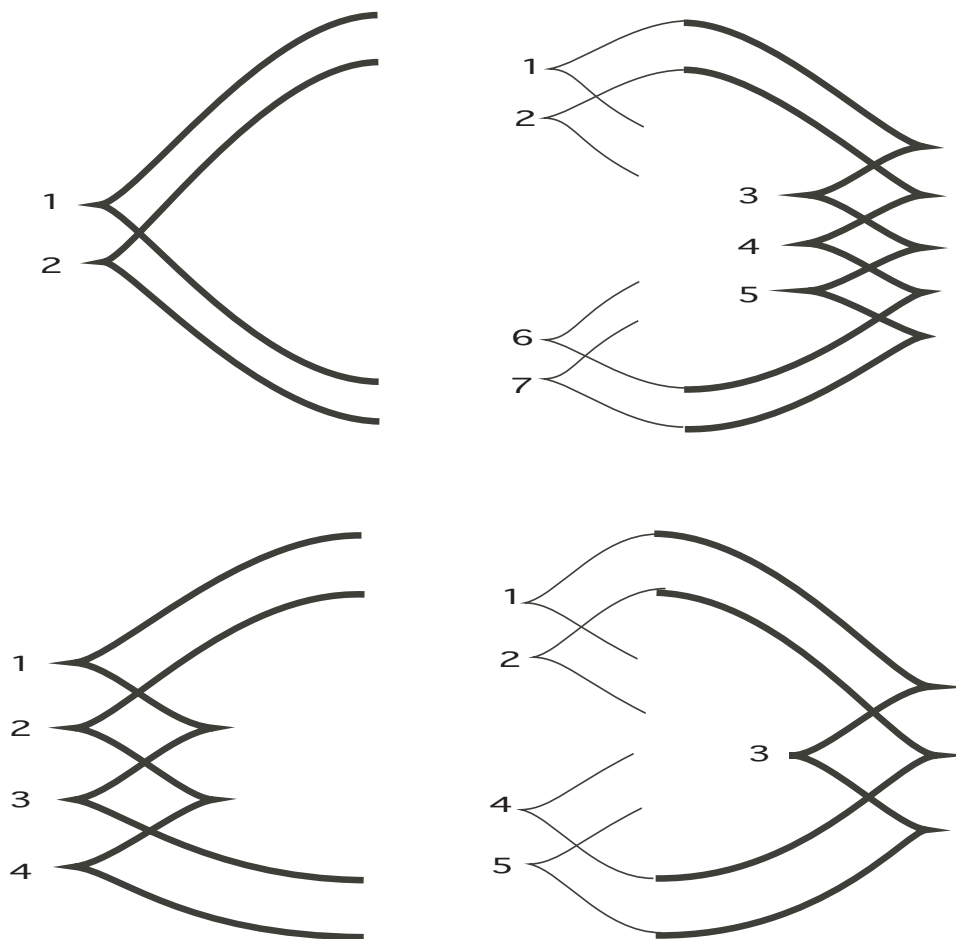


Figure 5.1: The two versions of $(-5, 2)$ with maximal Thurston-Bennequin invariant (see Figure 4.2) have been split in two in order to calculate the interval algebra. The borrowed cusps on the right hand side are shown with fine lines.

shown in figure 4.1, we know that given the right hand side of a knot we have for any value of n_1 there will be q left cusps that connect to the top q strands on the right hand side, and also q left cusps that connect to the bottom q strands on the right hand side. Note that when $n_1 = 0$, the same q left cusps will connect to the top and bottom of the right hand side. In Figure 5.1,

the interval algebra for the right hand side of the version of $(-5, 2)$ on the top is $I_2(7)$, and the interval algebra for the left hand side of the version of $(-5, 2)$ on the bottom is $I_2(5)$. For more details, see the section concerning $(-4, 3)$ below.

The double interval algebra is the free product of $I_q(L)$ and $I_q(R + q)$ with interval ends of length $q - 1$ identified. We denote this product by $\mathbb{I}_q(L, R + q) = \mathbb{I}_q((n_1 + 1)q, (n_2 + 2)q + e)$, and this is the algebra associated to the entire knot K . The differential defined above is still well-defined for the double interval algebra. When $L = q$, the algebra is equivalent to the circular algebra $O_q(R)$ which can be visualized as an algebra of arcs of length less than or equal to q on a circle with R marked points.

Remark 5.2.1. In Mishachev's paper, he claims that the double interval algebra is different than the one above for negative torus links. His formula is correct for negative torus links with unknotted components. \diamond

The above description of the full *DGA* of a knot projection will be needed to describe augmentations of a component of a torus link projection, and thus the possible augmentations of the projection of the entire link.

We now turn our attention to labeling of the 3-copy. When working with CH_{132} , we only need to look at vertices that are in $\mathbb{T}_{13}, \mathbb{T}_{32}$, and \mathbb{T}_{12} . Again, setting up a good labeling system will be essential. Start by labeling the left cusps of the link top to bottom as $1, 1, 1, 2, 2, 2, \dots, L, L, L$, starting on the left hand side of the diagram and continuing on the right hand side. Label

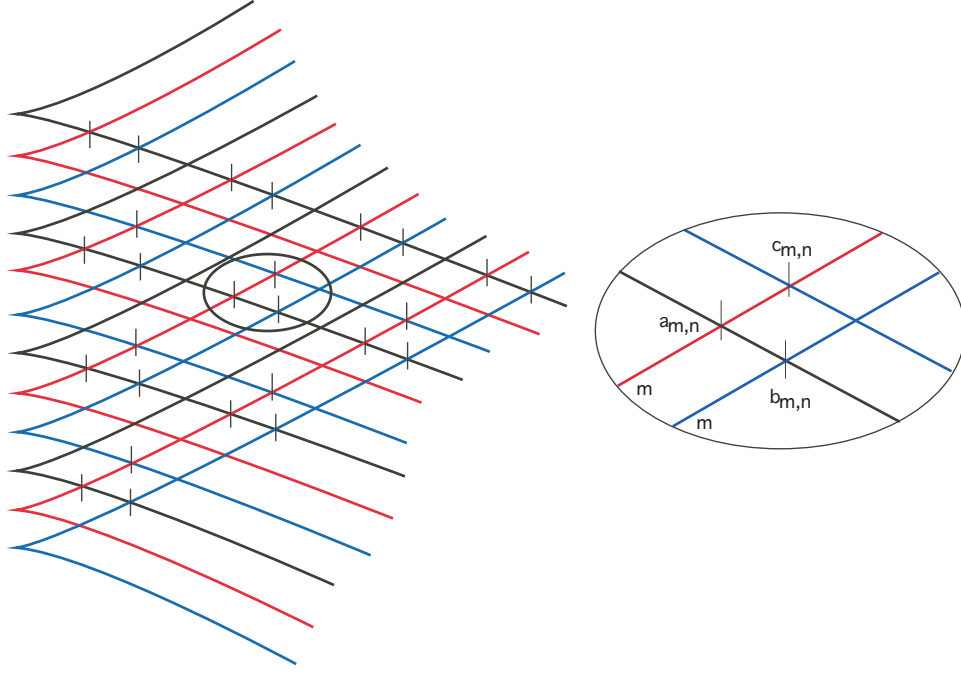


Figure 5.2: Labeling of the vertices that generate \mathbb{T}_{132} . In this example, $m = 3$ and $n = 2$. No augmentation is specified here.

the vertices of the projection of (p, q) (or $(2p, 2q)$) as shown in Figure 5.2. The vertex $a_{m,n} \in \mathbb{T}_{12}$ is at a crossing of the first (black) and second (red) components of the link, and occurs on the line with positive slope coming from the m^{th} (red) left cusp at the n^{th} time the first (black) component crosses that line. Similarly, the vertex $b_{m,n} \in \mathbb{T}_{13}$ occurs at a crossing of the first and third components of the link, and $c_{m,n} \in \mathbb{T}_{32}$ occurs at a crossing of the third (blue) and second (red) components of the link. Notice in Figure 5.2 that $a_{m,n}, b_{m,n}, c_{m,n}$ form a left pointing triangle opposite a right pointing triangle with two blue sides. Emanating from the m^{th} -triple of left cusps, we will have (m, n) -triangles for $n = 1, 2, \dots, \min(m, 2q)$. We will use the

convention that $a_{m,0}, b_{m,0}, c_{m,0} = 0$ for all m . Note that on the right hand side of the diagram, the m^{th} left cusp may be on the left hand side of the diagram. The identification coming from $\mathbb{I}_q(L, R + q)$ will be used to relabel those strands on the right hand side. This labeling can be seen explicitly in Figure 5.12 in the section concerning $(-4, 3)$. Notice in this example that the second index in $a_{m,n}, b_{m,n}, c_{m,n}$ will start on the left hand side of the diagram and continue onto the right hand side, i.e. if the left hand side ends with $a_{3,2}$ then the right hand side starts with $a_{3,3}$.

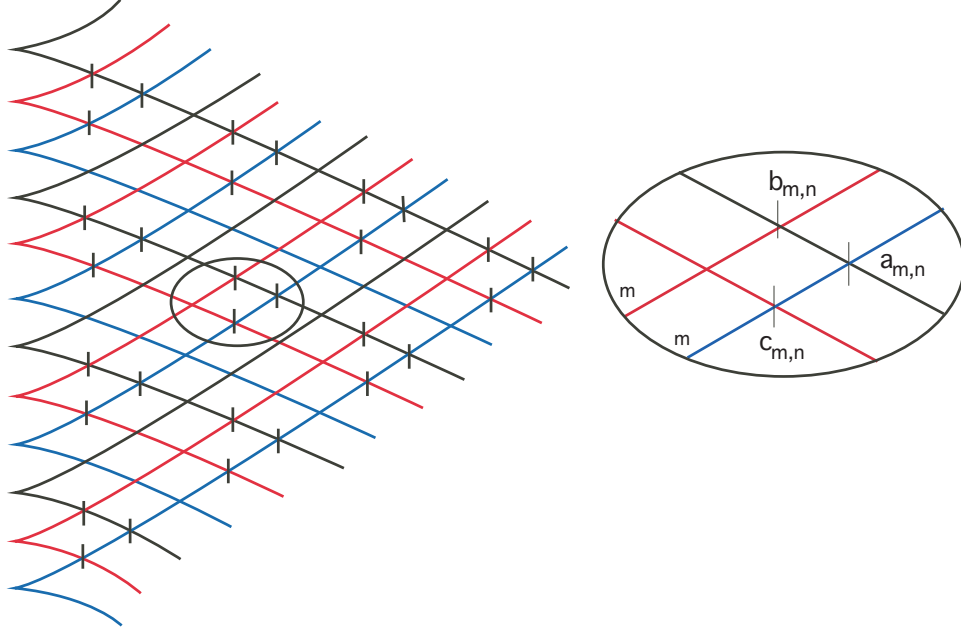


Figure 5.3: Labeling of the vertices that generate \mathbb{T}_{123} . In this example, $m = 3$ and $n = 2$. No augmentation is specified here.

When working with CH_{123} , we only need to look at vertices that are in $\mathbb{T}_{12}, \mathbb{T}_{23}$, and \mathbb{T}_{13} . As above, start by labeling the left cusps top to bottom

as $1, 2, \dots, L$, starting on the left hand side of the diagram and continuing on the right hand side. Label the vertices of (p, q) as shown in Figure 5.3. The vertex $a_{m,n} \in \mathbb{T}_{13}$ is at a crossing of the first (black) and third (blue) components of the link, and occurs on the line with positive slope coming from the m^{th} (blue) left cusp at the n^{th} time the first (black) component crosses that line. The vertex $b_{m,n} \in \mathbb{T}_{12}$ occurs at a crossing of the first (black) and second (red) components of the link, and $c_{m,n} \in \mathbb{T}_{23}$ occurs at a crossing of the second (red) and third (blue) components of the link. Notice in Figure 5.3 that $a_{m,n}, b_{m,n}, c_{m,n}$ form a right pointing triangle opposite a left pointing triangle with two red sides. As above, the identifications coming from $\mathbb{I}_q(L, R + q)$ are used to relabel the right hand side of the diagram.

5.3 Permutations of Negative $(p, 2)$ -Torus Links

Recall that there are $2m$ distinct Legendrian versions of an oriented negative $(p, 2)$ -torus knot. See Figure 4.2 for example.

Proposition 5.3.1. *The interval algebra associated to each version of a negative $(p, 2)$ -torus knot has a unique augmentation.*

Proof. The double interval algebra associated to $(p, 2)$ is $\mathbb{I}_2((n_1 + 1)2, (n_2 + 2)2 + 1)$ (here $e = 1$) as described in the previous section. The differential of

$\mathbb{I}_2((n_1 + 1)2, (n_2 + 2)2 + 1)$ is given by

$$\partial a_{i,j} = \begin{cases} 1 + a_{i,i+1}a_{i+1,j}, & j = i + 2 \\ 0, & j - i = 1. \end{cases}$$

Recall that an augmentation is a ring homomorphism $\epsilon : A \rightarrow \mathbb{Z}_2$ where $\epsilon \circ \partial = 0$. It is clear from the above formula for ∂ that it is necessary to augment every $a_{i,j}$ for $j - i = 1$, i.e. $\epsilon(a_{i,j}) = 1$ if and only if $j - i = 1$. Since these are all of the generators of the algebra, this augmentation must be unique. \square

Proposition 5.3.2. *Let K be a negative $(p, 2)$ torus knot with maximal Thurston-Bennequin invariant, and let ϵ be the unique augmentation of its associated double interval algebra. Both $CH_{123}(\epsilon)$ and $CH_{132}(\epsilon)$ have zero divisors, and so $z_{123} = z_{132}$.*

Proof. As shown above, a negative $(p, 2)$ -torus knot has a unique proper augmentation ϵ . From the diagram of L_{132} shown on the left of Figure 5.4, we can easily see that for all i

$$\partial_{132}^\epsilon(a_{i,1}) = \partial_{132}^\epsilon(b_{i,1}) = \partial_{132}^\epsilon(c_{i,1}) = 0$$

and

$$\partial_{132}^\epsilon(a_{i,2}) = a_{i,1} + a_{i-1,1} + b_{i-1,1}c_{i,1}$$

$$\partial_{132}^\epsilon(b_{i,2}) = b_{i,1} + b_{i-1,1}$$

$$\partial_{132}^\epsilon(c_{i,2}) = c_{i,1} + c_{i-1,1}.$$

Thus in CH_{132} , all of the $b_{i,1}$ terms are equivalent and all of the $c_{i,1}$ terms are equivalent, call these terms b and c respectively. Thus, for all i , we have $\partial_{132}^\epsilon(a_{i,2}) = a_{i,1} + a_{i-1,1} + bc$, and so $a_{i,1} + a_{i-1,1} \equiv bc$. Since this holds for all i , we have $a_{i,1} + a_{i-1,1} \equiv a_{i+1,1} + a_{i,1}$, and so $a_{i-1,1} \equiv a_{i+1,1}$. Thus on each side of the diagram, we have that all of the $a_{odd,1}$ terms are equivalent, and all of the $a_{even,1}$ terms are equivalent. Next we look at the algebra of the entire link, that is we make the identifications coming from the definition of the double interval algebra. We have that $a_{1,1}$ (on the left) is identified with $a_{1,1}$ (on the right). This means that all of the $a_{odd,1}$ terms are equivalent in $CH_{132}(\epsilon)$. We also have that $a_{2,1}$ (on the left) is identified with $a_{2,1}$ (on the right). This means that all of the $a_{even,1}$ terms are equivalent in $CH_{132}(\epsilon)$. But, looking at the other end of the interval, we have the identification $a_{(n_1+1)2,1}$ (on the left) $\leftrightarrow a_{(n_2+1)2+1,1}$ (on the right). Since $(n_1+1)2$ is always even and $(n_2+2)2+1$ is always odd for a negative $(p, 2)$ -torus knot, this means that *all* of the $a_{*,1}$ terms are equivalent in $CH_{132}(\epsilon)$. In particular,

$$\partial_{132}^\epsilon(a_{i,2}) = a_{i,1} + a_{i-1,1} + bc \equiv a_{i,1} + a_{i,1} + bc \equiv 0 + bc.$$

Recall that bc is a zero divisor of $CH_{132}(\epsilon)$ if $\partial_{132}^\epsilon(a) = bc$, and $b, c \notin \text{Im} \partial_{132}^\epsilon$ for $a \in \mathbb{T}_{12}$, $b \in \mathbb{T}_{13}$, $c \in \mathbb{T}_{23}$. By the above calculation of $\partial_{132}^\epsilon(a_{i,2})$, it is clear that bc is a zero divisor. Thus $CH_{132}(\epsilon)$ has zero divisors, and so $z_{132} = 1$.

From the diagram of L_{123} shown on the right of Figure 5.4, we can easily see that $\partial_{123}^\epsilon(a_{i,1}) = b_{i,1}c_{i,1}$. We also have that $\partial_{123}^\epsilon(b_{i,2}) = b_{i,1} + b_{i-1,1}$ and

$\partial_{123}^\epsilon(c_{i,2}) = c_{i,1} + c_{i-1,1}$. Thus all of the $b_{i,1}$ terms are equivalent and all of the $c_{i,1}$ terms are equivalent. There are no other relations. Hence CH_{123} has zero divisors and so $z_{123} = 1$. \square

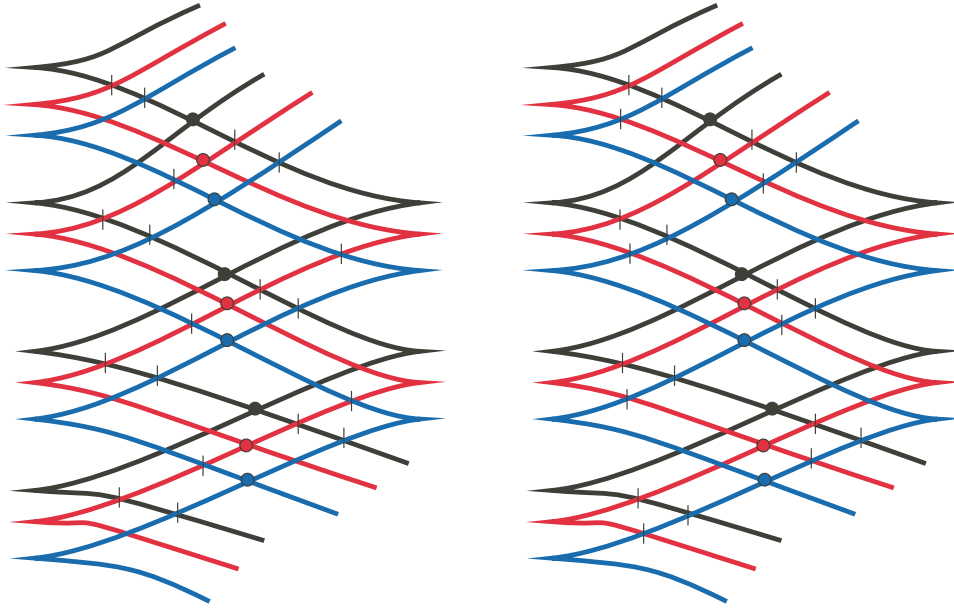


Figure 5.4: Each side of the triple of a Legendrian version of a negative $(p, 2)$ -torus link will look like the figures above. On the left is L_{132} and on the right is L_{123} . Augmented vertices are represented by dots.

In the proof above, notice that the algebra associated to each side of the knot does not have zero divisors in the CH_{132} case. It is only when the identifications coming from the definition of the double interval algebra are made that we get zero divisors. This happens precisely because the algebra is $\mathbb{I}_2((n_1 + 1)2, (n_2 + 2)2 + 1)$, and $(n_1 + 1)2$ is always even and $(n_2 + 2)2 + 1$ is always odd. As shown, the identifications make all of the

$a_{*,1}$ terms equivalent and so there are zero divisors. In Mishachev's proof for unknots, there was an even number of marked points on each side, and so there were still no zero divisors when the identification was made. If we double the knot, i.e. look at $2(p, 2)$, then we eliminate this problem in the above proof. *However*, the algebra associated to $2(p, 2)$ has many augmentations, and so we must do many more calculations. The procedure will be the same as for the $(-4, 3)$ example detailed in the next section. In fact, the algebra associated to $2(-p, 2)$ will have 27 “minimal augmentations” exactly as is shown for $(-4, 3)$.

5.4 Permutations of the $(-4, 3)$ -Torus Link

Conjecture 5.4.1. *Only cyclic permutations are possible for the n -copy of $(-4, 3)$.*

In this section we will work with a fixed projection of $(-4, 3)$ (see Figure 5.7) and use this projection to outline how it may be possible to show that noncyclic permutations are not possible of the n -copy of $(-4, 3)$. Since we are fixing a projection, we will call an augmentation of the algebra associated to the knot simply an *augmentation of the knot*. We will argue that only cyclic permutations are possible by working with a 6-copy of the knot, viewed as a 3-copy of the 2-copy, and show that the permutation 123 to 132 is not possible by associating a characteristic algebra to each of the two links (i.e. the link with strands ordered 123 and the link with strands ordered 132).

Note that if all permutations of the n -copy of the knot were possible, then all permutations of the 6-copy would be possible, specifically when viewed as a 3-copy of the 2-copy. Hence, if we can show that a noncyclic permutation is not possible for our 6-copy, then it must be the case that all permutations are not possible for the link.

Recall that our first step is to find the *DGA* of the knot. The interval algebra for the left side of $(-4, 3)$ is $I_3(3)$, with the correspondence shown in Figure 5.5. This algebra has generators $a_{1,2}, a_{2,3}, a_{1,3}$ and no units in its differential as the left side of this knot has no right cusps.

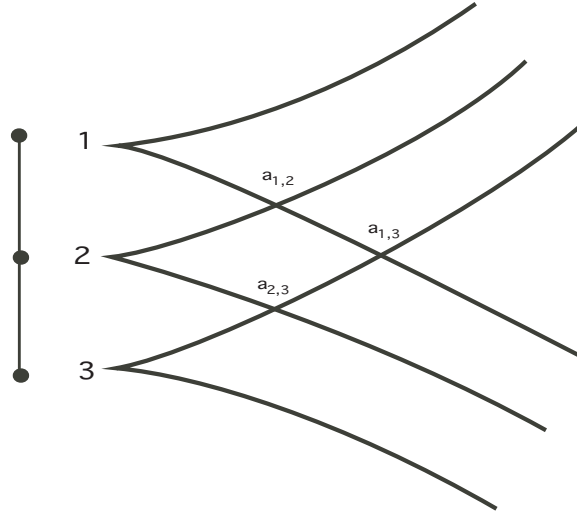


Figure 5.5: The left side of $(-4, 3)$ has interval algebra $I_3(3)$.

The right hand side of $(-4, 3)$ has interval algebra $I_3(7)$ as shown in Figure 5.6. In the figure, the right side of the knot is shown in bold lines, while the “borrowed” cusps from the left hand side are shown in fine lines.

This algebra has generators $a_{1,2}, a_{2,3}, a_{1,3}, a_{1,4}, a_{2,4}, a_{3,4}, a_{2,5}, a_{3,5}, a_{4,5}, a_{3,6}, a_{4,6}, a_{5,6}, a_{4,7}, a_{5,7}, a_{6,7}$.

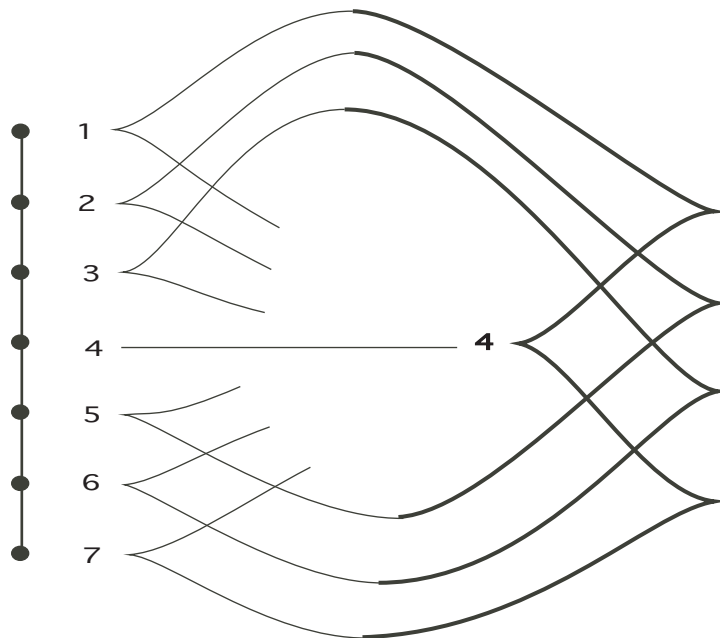


Figure 5.6: The right side of $(-4, 3)$ has interval algebra $I_3(7)$.

The algebra of $(-4, 3)$ is $\mathbb{I}_3(3, 7) = O_3(4)$ with the identifications $1 \leftrightarrow 5$, $2 \leftrightarrow 6$, and $3 \leftrightarrow 7$ as shown in Figure 5.7. Thus we have the identified vertices $a_{1,2} \equiv a_{5,6}$, $a_{2,3} \equiv a_{6,7}$, and $a_{1,3} \equiv a_{5,7}$ from the two halves shown in figures 5.5 and 5.6. This identification corresponds to the fact that the borrowed cusps on the top and bottom in figure 5.6 are the same cusps from the left hand side of the diagram, and hence are identified in the full knot.

The full differential of $\mathbb{I}_3(3, 7)$ is given by

$$\partial a_{1,2} = \partial a_{2,3} = \partial a_{3,4} = \partial a_{4,5} = 0$$

$$\partial a_{1,3} = a_{1,2}a_{2,3}$$

$$\partial a_{2,4} = a_{2,3}a_{3,4}$$

$$\partial a_{3,5} = a_{3,4}a_{4,5}$$

$$\partial a_{4,6} = a_{4,5}a_{5,6} = a_{4,5}a_{1,2}$$

$$\partial a_{1,4} = 1 + a_{1,2}a_{2,4} + a_{1,3}a_{3,4}$$

$$\partial a_{2,5} = 1 + a_{2,3}a_{3,5} + a_{2,4}a_{4,5}$$

$$\partial a_{3,6} = 1 + a_{3,4}a_{4,6} + a_{3,5}a_{5,6} = 1 + a_{3,4}a_{4,6} + a_{3,5}a_{1,2}$$

$$\partial a_{4,7} = 1 + a_{4,5}a_{5,7} + a_{4,6}a_{6,7} = 1 + a_{4,5}a_{1,3} + a_{4,6}a_{2,3}$$

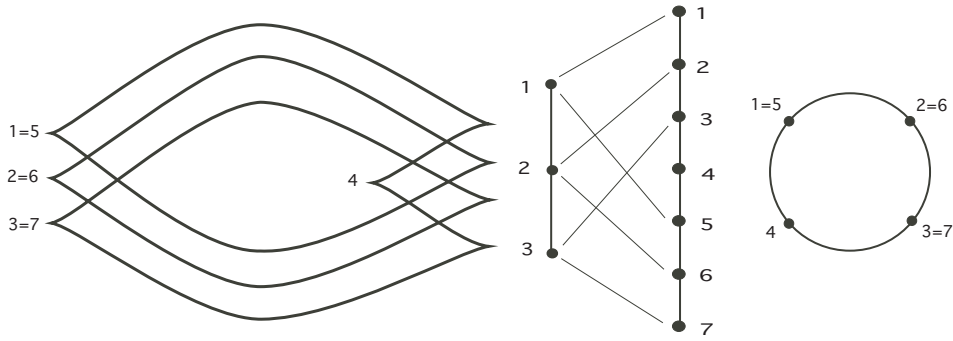


Figure 5.7: $(-4, 3)$ has double interval algebra $\mathbb{I}_3(3, 7) = O_3(4)$.

Now that we have found the *DGA*, the next step is to look for augmentations of the algebra.

Proposition 5.4.1. *The algebra $\mathbb{I}_3(3, 7) = O_3(4)$ associated to this projection of the $(-4, 3)$ -torus knot does not have a proper augmentation.*

Proof. From the calculation of $\partial a_{1,4}$ above, we see that we must augment either the pair $a_{1,2} \equiv a_{5,6}$ and $a_{2,4}$, or $a_{1,3}$ and $a_{3,4}$, but not all four. If we augment $a_{1,2}$ and $a_{2,4}$, then we cannot augment $a_{2,3}$ or $a_{4,5}$ as this creates a 1 in the differential of $a_{1,3}$ or $a_{4,6}$. But now we cannot remove the 1 in $\partial a_{4,7}$, and so the augmentation cannot be completed. If instead we augment $a_{1,3}$ and $a_{3,4}$, then again we cannot augment $a_{2,3}$ or $a_{4,5}$, and so the augmentation cannot be completed. Thus, there is no augmentation of $\mathbb{I}_3(3, 7)$. \square

It was shown above that $\mathbb{I}_3(3, 7)$ has no proper augmentation. Proposition 2.3.2 guarantees that the double of $(-4, 3)$ will have at least one proper augmentation. The algebra associated to $2(-4, 3)$ is $\mathbb{I}_6(6, 14)$ (to see this, follow the set up for $(-4, 3)$ above) and has identifications $1 \leftrightarrow 9$, $2 \leftrightarrow 10$, and $3 \leftrightarrow 11$, $4 \leftrightarrow 12$, $5 \leftrightarrow 13$, and $6 \leftrightarrow 14$. As a step towards finding all of the augmentations, we first find all of the augmentations that have a single pair of generators augmented in each $\partial a_{i,j}$ term when $j - i = 6$ and no other generators augmented. Note that this means that we will be augmenting a minimal number of generators, which is 8 (the number of right cusps of $2(-4, 3)$) in the case of $\mathbb{I}_6(6, 14)$. Call such an augmentation a *minimal augmentation*.

Remark 5.4.1. It is important to note that while augmenting only these pairs does give a proper augmentation of the algebra, there can be other augmentations as well that contain the sets of augmented generators determined in the following lemma.

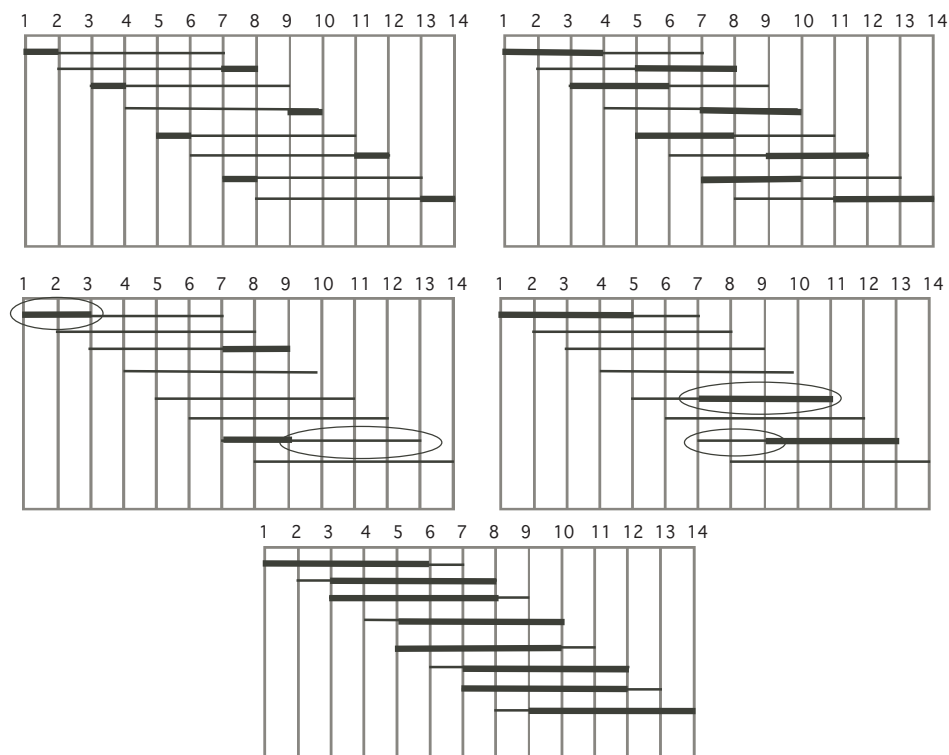
Lemma 5.4.1. $\mathbb{I}_6(6, 14)$ has three minimal augmentations, each containing 8 augmented vertices.

Proof. When $j - i = 6$, $a_{i,j}$ is a right cusp and we have

$$\partial a_{i,j} = 1 + a_{i,i+1}a_{i+1,j} + a_{i,i+2}a_{i+2,j} + a_{i,i+3}a_{i+3,j} + a_{i,i+4}a_{i+4,j} + a_{i,i+5}a_{i+5,j}.$$

Note that the choice of augmentation pair in $\partial a_{1,7}$ will determine all of the other augmentation pairs for the knot. These choices are $a_{1,i}$ where $1 < i \leq q$. In the argument below, we will show which of these choices of $a_{1,i}$ give a proper augmentation of the algebra, and hence of the knot. There are five choices for $a_{1,i}$ in this case. Augmenting any of $a_{1,2}$, $a_{1,4}$, $a_{1,6}$ leads to a proper augmentation of the knot as shown with intervals in figure 5.8. If we augment $a_{1,3} \equiv a_{9,11}$, then we must augment $a_{3,7}$ and then $a_{7,9}$. Augmenting $a_{7,9}$ means that we must augment $a_{9,13} \equiv a_{1,5}$. Since we are assuming that we are only augmenting one generator of the form $a_{1,i}$ this gives us a contradiction. If we augment $a_{1,5} \equiv a_{9,13}$, then we must augment $a_{5,7}$ and then $a_{7,9}$. Augmenting $a_{5,7}$ means that we must augment $a_{7,11}$. Since we are assuming that we are only augmenting one generator of the form $a_{7,i}$ this gives us a contradiction. Thus $\mathbb{I}_6(6, 14)$ has three choices for augmentations with only 8 augmented generators, i.e. three minimal augmentations. \square

As shown above, the double of $(-4, 3)$ has three augmentations with only 8 augmented vertices. We will label these augmentations A , B , and C . The


 Figure 5.8: Possible augmentations for $\mathbb{I}_6(6, 14)$

augmented vertices of each are given below.

$$(A) \ a_{1,2}, a_{3,4}, a_{5,6}, a_{7,8}, a_{2,7}, a_{4,1}, a_{6,3}, a_{8,5}$$

$$(B) \ a_{2,3}, a_{4,5}, a_{1,6}, a_{6,7}, a_{3,1}, a_{5,3}, a_{7,5}, a_{8,1}$$

$$(C) \ a_{1,4}, a_{2,5}, a_{3,6}, a_{4,7}, a_{5,8}, a_{6,1}, a_{7,2}, a_{8,3}.$$

Augmentation A is represented by the solid black dots in figure 5.9. Augmentation B is represented by the open dots in figure 5.9. Augmentation C is represented by the gray dots in figure 5.9. The triple of $2(-4, 3)$ will then have 27 minimal augmentations, $AAA, AAB, AAC, \dots, CCC$.

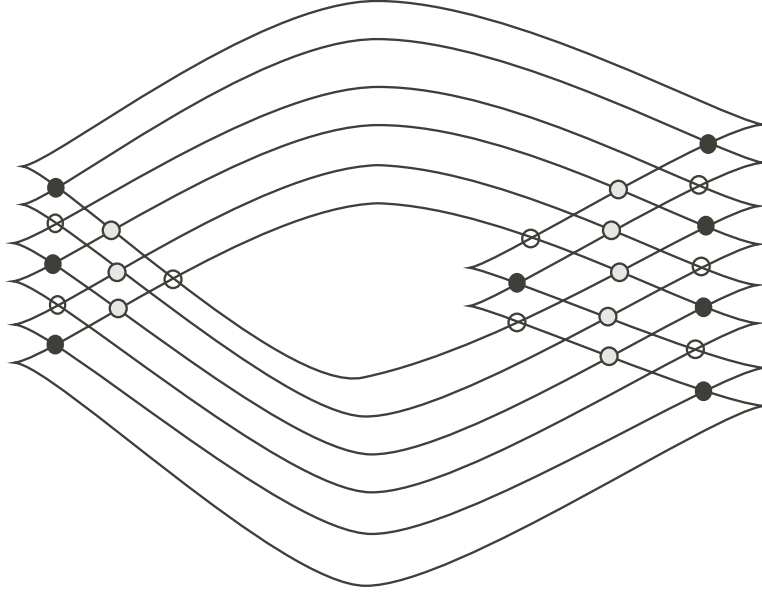


Figure 5.9: Double of $(-4, 3)$ with its three minimal augmentations. Augmented vertices are represented by dots.

Lemma 5.4.2. *For every i, j such that $j - i = 6$, any augmentation of $2(-4, 3)$ can only have one pair of the form $a_{i,k}a_{k,j}$ augmented for $1 \leq k \leq 5$.*

Before giving the proof of this lemma, note that this means that any augmentation of $2(-4, 3)$ will be one of the minimal augmented vertices with (perhaps) added non-paired vertices. When we add vertices to A , B , and C we get new augmentations $A = A_1, A_2, \dots, A_k$; $B = B_1, B_2, \dots, B_m$; and $C = C_1, C_2, \dots, C_n$ for $k, m, n \geq 1$.

Proof. Recall that

$$\partial a_{i,j} = 1 + a_{i,i+1}a_{i+1,j} + a_{i,i+2}a_{i+2,j} + a_{i,i+3}a_{i+3,j} + a_{i,i+4}a_{i+4,j} + a_{i,i+5}a_{i+5,j}.$$

In order to have an augmentation, as far as pairs go, we must augment (1) all five pairs, (2) exactly three pairs, or (3) exactly one pair.

It can be shown directly from the ∂ calculation that (1) and (2) are impossible. To see that (1) is not possible, fix a generator of the form $a_{i,j}$ where $j - i = 6$. In this proof, we will use $a_{1,7}$. Because of the symmetry in the calculations, any other generator of this form will follow the same argument. Suppose that all five pairs of generators in $\partial a_{1,7}$ are augmented:

$$\partial a_{1,7} = 1 + a_{1,2}a_{2,7} + a_{1,3}a_{3,7} + a_{1,4}a_{4,7} + a_{1,5}a_{5,7} + a_{1,6}a_{6,7}.$$

Then it is not possible to augment $a_{2,3}$, $a_{5,6}$, $a_{7,8}$, or $a_{8,9}$ as this will create a unit in the differential calculations of $a_{1,3}$, $a_{5,7}$, $a_{6,8}$, and $a_{8,10}$, respectively. Now $\partial a_{7,10} = a_{7,8}a_{8,10} + a_{7,9}a_{9,10}$ implies that $a_{7,9}$ cannot be augmented since $a_{7,8}$ is not and $a_{1,2} = a_{9,10}$ is. Now look at

$$\partial a_{3,7} = a_{3,4}a_{4,7} + a_{3,5}a_{5,7} + a_{3,6}a_{6,7}$$

and

$$\partial a_{3,9} = 1 + a_{3,4}a_{4,9} + a_{3,5}a_{5,9} + a_{3,6}a_{6,9} + a_{3,7}a_{7,9} + a_{3,8}a_{8,9}.$$

In the first equation, since $a_{4,7}$, $a_{5,7}$, and $a_{6,7}$ are all augmented, we must either augment none of $a_{3,4}$, $a_{3,5}$, and $a_{3,6}$ or exactly two of them. In the second equation, since $a_{7,9}$ and $a_{8,9}$ are not augmented, we must augment at least one of $a_{3,4}$, $a_{3,5}$, $a_{3,6}$. Therefore, we must augment exactly two of them.

There are three choices: $a_{3,4}$ and $a_{3,5}$, $a_{3,4}$ and $a_{3,6}$, or $a_{3,5}$ and $a_{3,6}$. Suppose we augment $a_{3,4}$ and $a_{3,5}$, but not $a_{3,6} \equiv a_{11,14}$. Augmenting $a_{3,4}$ means that it is not possible to augment $a_{4,5}$ as this would create a unit in the differential of $a_{3,5}$. Now $\partial a_{4,7} = a_{4,5}a_{5,7} + a_{4,6}a_{6,7}$, so we can't augment $a_{4,6} \equiv a_{12,14}$ since $a_{5,7}$ and $a_{6,7}$ are augmented and $a_{4,5}$ is not. We now must augment $a_{2,4}$ because $\partial a_{1,4} = a_{1,2}a_{2,4} + a_{1,3}a_{3,4}$ and $a_{2,5}$ because $\partial a_{1,5} = a_{1,2}a_{2,5} + a_{1,3}a_{3,5} + a_{1,4}a_{4,5}$. Augmenting $a_{2,4}$ and $a_{2,5}$ means that we cannot augment $a_{2,6} \equiv a_{10,14}$ since $\partial a_{2,7} = a_{2,3}a_{3,7} + a_{2,4}a_{4,7} + a_{2,5}a_{5,7} + a_{2,6}a_{6,7}$. But now we cannot eliminate the unit in the calculation

$$\partial a_{8,14} = 1 + a_{8,9}a_{9,14} + a_{8,10}a_{10,14} + a_{8,11}a_{11,14} + a_{8,12}a_{12,14} + a_{8,13}a_{13,14}.$$

Hence it is not possible to augment $a_{3,4}$ and $a_{3,5}$, but not $a_{3,6}$. Now instead try to augment $a_{3,4}$ and $a_{3,6}$, but not $a_{3,5} \equiv a_{11,13}$. Then we cannot augment $a_{4,5} \equiv a_{12,13}$ as above. We then cannot augment $a_{2,5} \equiv a_{10,13}$ since $\partial a_{1,5} = a_{1,2}a_{2,5} + a_{1,3}a_{3,5} + a_{1,4}a_{4,5}$. But now we cannot eliminate the unit in the calculation

$$\partial a_{7,13} = 1 + a_{7,8}a_{8,13} + a_{7,9}a_{9,13} + a_{7,10}a_{10,13} + a_{7,11}a_{11,13} + a_{7,12}a_{12,13}.$$

So it is not possible to augment $a_{3,4}$ and $a_{3,6}$, but not $a_{3,5}$. Finally, we try to augment $a_{3,5}$ and $a_{3,6}$, but not $a_{3,4}$. We then cannot augment $a_{2,4}$ since $\partial a_{1,4} = a_{1,2}a_{2,4} + a_{1,3}a_{3,4}$. Because $\partial a_{2,7} = a_{2,3}a_{3,7} + a_{2,4}a_{4,7} + a_{2,5}a_{5,7} +$

$a_{2,6}a_{6,7}$, we must augment both of $a_{2,5}$ and $a_{2,6}$ or neither of them. But

$$\partial a_{2,8} = 1 + a_{2,3}a_{3,8} + a_{2,4}a_{4,8} + a_{2,5}a_{5,8} + a_{2,6}a_{6,8} + a_{2,7}a_{7,8}$$

tells us that we must augment at least one of $a_{2,5}$ or $a_{2,6}$. Therefore, we must augment both. This means that we can only augment one of $a_{5,8}$ or $a_{6,8}$ in order to eliminate the unit in $\partial a_{2,8}$. But this will create a unit in

$$\partial a_{3,8} = a_{3,4}a_{4,8} + a_{3,5}a_{5,8} + a_{3,6}a_{6,8} + a_{3,7}a_{7,8}.$$

Therefore, it is not possible to augment $a_{3,5}$ and $a_{3,6}$, but not $a_{3,4}$. Since one of these possibilities had to be true in order to complete the augmentation, we have that it is not possible to augment all five pairs of generators in $\partial a_{1,7}$. A similar argument will show that (1) is not possible for the other $\partial a_{i,j}$ terms when $j - i = 6$.

The proof that (2) is not possible is similar in style to the one above, but more complicated as all possible combinations of three pairs of generators must be checked. The proof will not be repeated here.

Therefore, it must be the case that there will be only one augmented pair in the ∂ calculation of the right cusps, i.e. the $\partial a_{i,j}$ terms when $j - i = 6$. \square

The above result will be important in our search for zero divisors in $CH_{132}(\epsilon)$. We first give the following lemma concerning zero divisors in $CH_{123}(\epsilon)$ that holds for any augmentation of the knot, not just the minimal

augmentations.

Lemma 5.4.3. *CH_{123} has zero divisors when all three components of the 3-copy of $(2(-4, 3))$ have the same augmentation.*

Proof. Label the left cusps of $2(-4, 3)$ top to bottom as $1, 2, \dots, 8$, and the vertices of the 6-copy of $(-4, 3)$ as shown in Figure 5.3. Because we are working with CH_{123} , we only need to look at vertices that are in $\mathbb{T}_{12}, \mathbb{T}_{23}$, and \mathbb{T}_{13} .

Note that for all j , $\partial_{123}(a_{j,1}) = b_{j,1}c_{j,1}$. We will now show that for a fixed k , $b_{k,1}, c_{k,1} \notin \text{Im } \partial_{123}$. Since ∂_{123} preserves the link grading, it suffices to show that there does not exist an $x \in \mathbb{T}_{12}$ such that $\partial_{123}(x) = b_{k,1}$. Note that, if x exists then $x = \sum b_{i,j}$. First notice that if no vertex of the form $b_{k+i,1+i}$ (vertices on the black strand of the $b_{k,1}$ crossing) or $b_{k,j}$ (vertices on the red strand of the $b_{k,1}$ crossing) is augmented for any value of i or j , then $b_{k,1}$ does not appear in the image of ∂_{123} and thus is nonzero. Now consider the case that $b_{k,1}$ does appear in the image ∂_{123} . First suppose that $x = b_{m,n}$ and so $b_{k,1} = \partial_{123}(b_{m,n})$; it follows that there exists an $m, n > 1$ so that there is a parallelogram with corners at $b_{m,n}$, an augmented vertex, and $b_{k,1}$, i.e., $\partial_{123}(b_{m,n}) = b_{k,1}$. This means that a portion of the diagram will look like the one in Figure 5.10. Since we are assuming that all three components have the same augmentation, the vertex directly below the augmented vertex in Figure 5.10 above must also be augmented. Therefore we get another term in the differential of $b_{m,n}$. In particular, if the m, j vertex is augmented, we

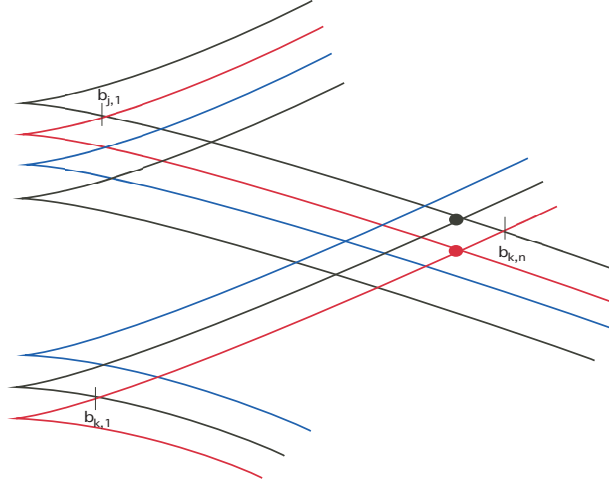


Figure 5.10: If the vertex at the intersection of the two black strands is augmented (the black dot), then the vertex at the intersection of the two red strands directly below it (the red dot) must also be augmented. This shows that it is not possible to have $b_{k,1}$ as the only summand in $\partial_{123}(b_{m,n})$

have that $\partial b_{m,n}$ has $b_{k,1}$, and $b_{k-j,1}$ as two of its summands.

Now suppose that $x = \sum b_{i,j}$. Using the argument above, we have that vertices of the form $b_{*,1}$ will always occur in summand pairs in the image of ∂_{123} . Hence $\partial_{123}(x)$ cannot be equal to $b_{k,1}$ alone, i.e. $b_{k,1} \neq 0$. A similar argument shows that $c_{k,1} \neq 0$. Since this argument is independent of which augmentation we use, we have that CH_{123} has zero divisors whenever all three components have the same augmentation. \square

As a step towards showing that CH_{132} has no zero divisors, we will first show that if $\partial_{132}a_{m,n}$ has no linear terms, then all terms of $\partial_{132}a_{m,n}$ are equivalent to zero in $CH_{132}(\epsilon)$ unless all three components have augmentation C described above. We then conjecture that in all but the CCC case, $CH_{132}(\epsilon)$

has no zero divisors.

As noted above, the three minimal augmentations for each component translate into 27 possible minimal augmentations of the link. We will give general formulas for ∂_{132} that include the extra variable $x_{i,j}$ that will be either 0 or 1 depending on the choice of augmentation. The labeling of each $x_{i,j}$ will be the same as for the $a_{i,j}$, $b_{i,j}$, and $c_{i,j}$ terms described in the previous section. The superscript on each $x_{i,j}$ term in the calculations below indicates which component the augmented vertex is on.

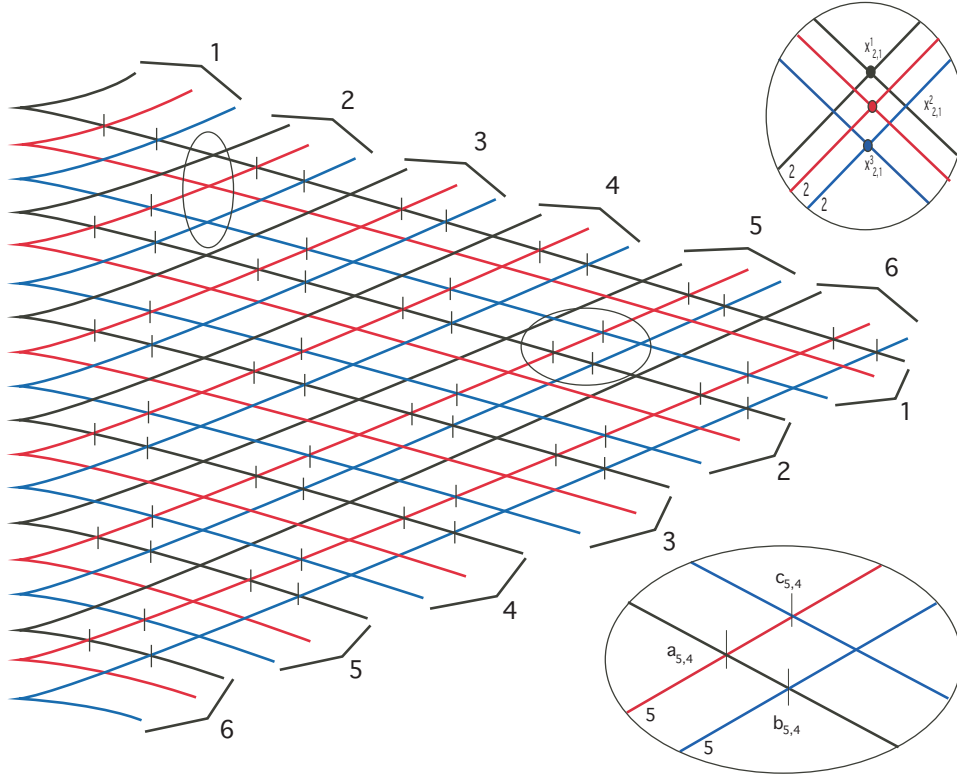
The three possible minimal augmentations translate into three possibilities for each component in terms of $x_{i,j}$. The three cases are:

$$(A) \ x_{2,1} = x_{4,1} = x_{6,1} = x_{8,1} = x_{7,5} = x_{9,5} = x_{11,5} = x_{13,5} = 1 \text{ and all other } x_{i,j} = 0$$

$$(B) \ x_{3,1} = x_{5,1} = x_{6,5} = x_{7,1} = x_{9,1} = x_{8,5} = x_{10,5} = x_{12,5} = 1 \text{ and all other } x_{i,j} = 0$$

$$(C) \ x_{4,3} = x_{5,3} = x_{6,3} = x_{7,3} = x_{8,3} = x_{9,3} = x_{10,3} = x_{11,3} = 1 \text{ and all other } x_{i,j} = 0$$

Lemma 5.4.4. *For every augmentation ϵ of $2(-4, 3)$, define $Q_{12}^\epsilon \subset \mathbb{T}_{12}$ to be the subspace generated by $a \in \mathbb{T}_{12}$ such that $\partial_{132}^\epsilon(a)$ has no linear terms. If $a \in Q_{12}^\epsilon$ is a generator of \mathbb{T}_{12} , and $\epsilon \neq CCC$ is a minimal augmentation, then $a = \sum b_i c_i$, where $b_i \in \text{Im} \partial_{132}^\epsilon$ or $c_i \in \text{Im} \partial_{132}^\epsilon$ for all i .*


 Figure 5.11: Left hand side of $6(-4, 3)$

In the following proof, for readability we will sometimes write “ $b_{m,n} = 0$ ” to mean $b_{m,n} \in \text{Im} \partial_{132}^\epsilon$, and so $b_{m,n} \equiv 0$ in CH_{132} .

Proof. Because we are working with CH_{132} , we only need to look at vertices that are in \mathbb{T}_{13} , \mathbb{T}_{32} , and \mathbb{T}_{12} .

We can easily see from figures 5.11 and 5.12 that $\partial_{132} a_{m,1} = \partial_{132} b_{m,1} = \partial_{132} c_{m,1} = 0$ for all m . When $n = 2$, we can compute from the diagram the

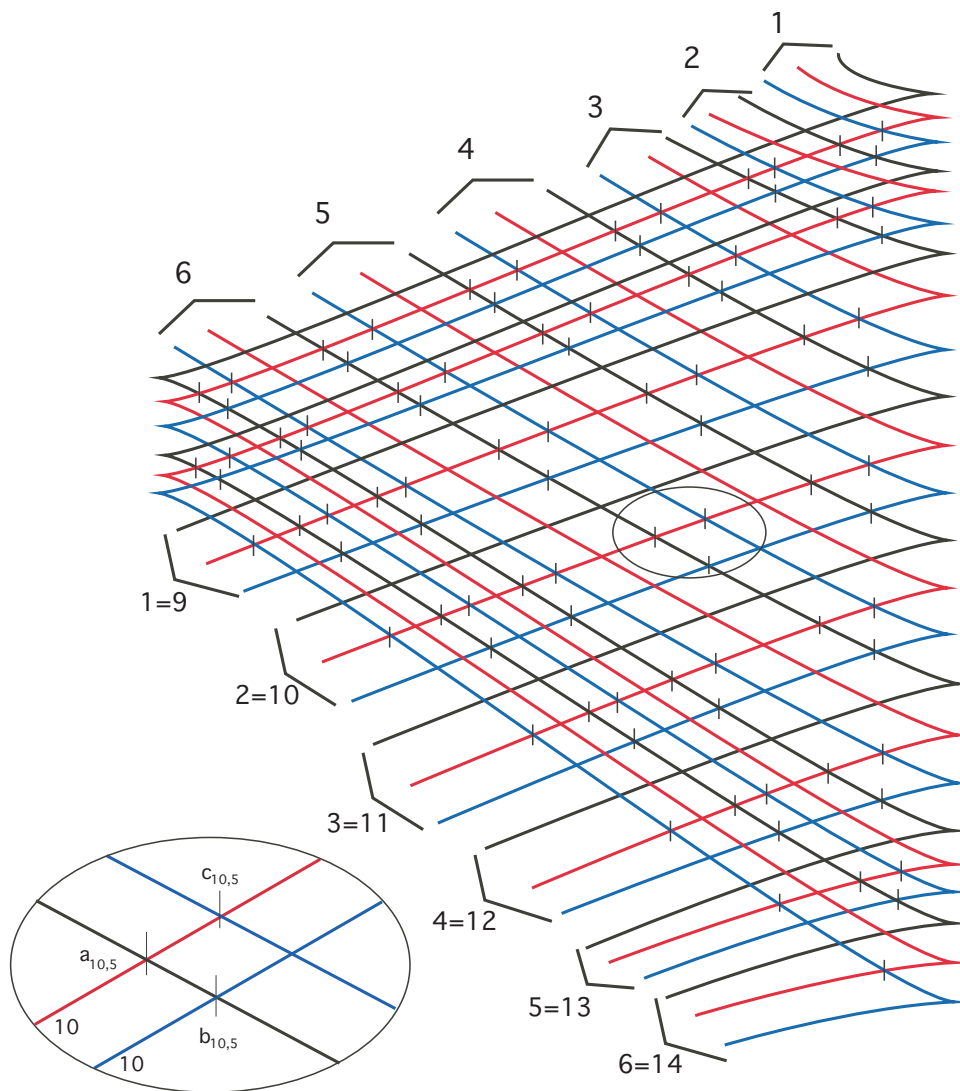


Figure 5.12: Right hand side of $6(-4, 3)$

∂_{132} map for all three types of vertices as follows:

$$\partial_{132}a_{m,2} = x_{m,1}^1 a_{m,1} + x_{m,1}^2 a_{m-1,1} + b_{m-1,1} c_{m,1}$$

$$\partial_{132}b_{m,2} = x_{m,1}^1 b_{m,1} + x_{m,1}^3 b_{m-1,1}$$

$$\partial_{132}c_{m,2} = x_{m-1,1}^3 c_{m,1} + x_{m,1}^2 c_{m-1,1}$$

We want to show that every summand of $\partial_{132}a_{m,2}$ is equivalent to zero in CH_{132} if $\partial_{132}a_{m,2}$ does not have any linear terms. Since every calculation will be the same, we will show this to be true for $\partial_{132}a_{4,2}$ for ease of reading. Assume that $\partial_{132}a_{4,2}$ has no linear terms. Then $x_{4,1}^1 = x_{4,1}^2 = 0$, and we know that the first and second component do not have augmentation A above. We want to show that either $b_{3,1} \in \text{Im}\partial_{132}$ or $c_{4,1} \in \text{Im}\partial_{132}$. There are 12 possible minimal augmentations to check: $BBA, CBA, BCA, CCA, BCA, CCA, BBB, CBB, BCB, CCB, BCC$, and CCC . We have that $\partial_{132}b_{4,2} = x_{4,1}^1 b_{4,1} + x_{4,1}^3 b_{3,1} = 0 + x_{4,1}^3 b_{3,1}$. If the third component has augmentation A then $x_{4,1}^3 = 1$ and so $b_{3,1} \in \text{Im}\partial_{132}$ and we are done. There now remain 6 minimal augmentations to check: BBB, CBB, BCB, CCB, BCC , and CCC . If $x_{4,1}^3 = 0$, then $\partial_{132}c_{5,2} = x_{4,1}^3 c_{5,1} + x_{5,1}^2 c_{4,1} = 0 + x_{5,1}^2 c_{4,1}$. If the second component has augmentation B then $x_{5,1}^2 = 1$ and so $c_{4,1} \in \text{Im}\partial_{132}$ and we are done. Otherwise, we now have that the second component has augmentation C described above. This leaves us with 4 augmentations to check: BCB, CCB, BCC , and CCC . Now $\partial_{132}c_{4,2} = x_{3,1}^3 c_{4,1} + x_{4,1}^2 c_{3,1} = x_{3,1}^3 c_{4,1} + 0$. If the third component has augmentation B then $x_{3,1}^3 = 1$ and so $c_{4,1} \in \text{Im}\partial_{132}$ and we are done. Otherwise, we now have that the third

component has augmentation C described above. This leaves only BCC and CCC to check. Now $\partial_{132}b_{3,2} = x_{3,1}^1b_{3,1} + x_{3,1}^3b_{2,1} = x_{3,1}^1b_{3,1} + 0$. If the first component has augmentation B then $x_{3,1}^1 = 1$ and so $b_{3,1} \in \text{Im}\partial_{132}$ and we are done. Otherwise, the first component has augmentation C . Thus we have shown that all terms of $\partial_{132}a_{4,2}$ are equivalent to zero in CH_{132} unless all three components have the augmentation C .

When $n = 3$, we have the following from the diagram:

$$\begin{aligned}\partial_{132}a_{m,3} &= x_{m-1,1}^1a_{m,2} + x_{m,1}^2a_{m-1,2} + b_{m-1,2}c_{m,1} + b_{m-2,1}c_{m,2} \\ \partial_{132}b_{m,3} &= x_{m-1,1}^1b_{m,2} + x_{m,1}^3b_{m-1,2} \\ \partial_{132}c_{m,3} &= x_{m-2,1}^3c_{m,2} + x_{m,1}^2c_{m-1,2}\end{aligned}$$

We will show that $\partial_{132}a_{m,3} = 0$ if it does not have linear terms by illustrating using $a_{5,3}$. If $\partial_{132}a_{5,3}$ has no linear terms, then $x_{4,1}^1 = x_{5,1}^2 = 0$, and we have that the first component does not have augmentation A and the second component does not have augmentation B . $\partial_{132}b_{5,3} = x_{4,1}^1b_{5,2} + x_{5,1}^3b_{4,2} = 0 + x_{5,1}^3b_{4,2}$. If the third component has augmentation B , then $b_{4,2} = 0$. In this case, we also have that $\partial_{132}c_{5,3} = x_{3,1}^3c_{5,2} + x_{5,1}^2c_{4,2} = c_{5,2} + 0$ and so $c_{5,2} = 0$ and we are done. Otherwise, we have that $\partial_{132}c_{5,2} = x_{4,1}^3c_{5,1} + x_{5,1}^2c_{4,1} = x_{4,1}^3c_{5,1} + 0$ and that $\partial_{132}b_{4,2} = x_{4,1}^1b_{4,1} + x_{4,1}^3b_{3,1} = 0 + x_{4,1}^3b_{3,1}$. Therefore, if $x_{4,1}^3 = 1$ (meaning that the third component has augmentation A) then both $b_{3,1}$ and $c_{5,1}$ are equal to 0 and we are done. Otherwise, we have that the third component has augmentation C . In this case, we get that $\partial_{132}b_{4,3} = x_{3,1}^1b_{4,2} + x_{4,1}^3b_{3,2} = x_{3,1}^1b_{4,2} + 0$. Thus if the first component

has augmentation B , we have that $b_{4,2} = 0$. In this case we also get that $\partial_{132}b_{3,2} = x_{3,1}^1b_{3,1} + x_{3,1}^3b_{2,1} = b_{3,1}$, so $b_{3,1}$ is also zero and we are done. Otherwise, we must have that the first component has augmentation C . In this case, we have that $\partial_{132}c_{6,3} = x_{4,1}^3c_{6,2} + x_{6,1}^2c_{5,2} = 0 + x_{6,1}^2c_{5,2}$ and so if the second component has augmentation A , we have that $c_{5,2} = 0$. We also get that $\partial_{132}c_{6,2} = x_{5,1}^3c_{6,1} + x_{6,1}^2c_{5,1} = c_{5,1}$ and so $c_{5,1} = 0$ and we are done. Otherwise, we now have that the second component also has augmentation C . We have now shown that $\partial_{132}a_{5,3} = 0$ unless all three components have augmentation C .

When $n \geq 4$, and $\partial_{132}a_{m,n}$ has no linear terms, then there will be only one choice for the augmentation on both the first and second components. As an example, the linear part of ∂_{132} is given by $\partial_{132}a_{6,5} = x_{5,3}^1a_{6,2} + x_{3,1}^1a_{6,4} + x_{6,3}^2a_{3,2} + x_{6,1}^2a_{5,4}$. If $\partial_{132}a_{6,5}$ has no linear terms, then the first component must have augmentation A described above, and the second component must have augmentation B described above. Knowing the specific augmentations on those two components makes the calculation showing $\partial_{132}a_{6,5} = 0$ much simpler. The quadratic terms of $\partial_{132}a_{6,5}$ are $b_{5,4}c_{6,1} + b_{4,3}c_{6,2} + b_{3,2}c_{6,3} + b_{2,1}c_{6,4}$. To see that the last term is 0, we compute $\partial_{132}b_{2,2}$. Now $\partial_{132}b_{2,2} = x_{2,1}^1b_{2,1} + x_{2,1}^3b_{1,1} = b_{2,1} + x_{2,1}^3b_{1,1}$, so if $x_{2,1}^3 = 0$ then $b_{2,1} = 0$. If $x_{2,1}^3 = 1$, then $\partial_{132}c_{6,5} = c_{6,4}$. Thus the last quadratic term of $\partial_{132}a_{6,5} = 0$ for either value of $x_{2,1}^3$. Similarly, to see that the third term is equal to 0 we compute that $\partial_{132}b_{3,3} = x_{2,1}^1b_{3,2} + x_{3,1}^3b_{2,2} = b_{3,2} + x_{3,1}^3b_{2,2}$, so if $x_{3,1}^3 = 0$ then $b_{3,2} = 0$. If $x_{3,1}^3 = 1$, then $\partial_{132}c_{6,4} = c_{6,3}$. Thus the third quadratic term of $\partial_{132}a_{6,5} = 0$ for

either value of $x_{2,1}^3$. The other two terms are shown using a similar argument.

We have now shown that unless all three components have augmentation C described above, that $\partial_{132}a_{m,n}$ has all terms equivalent to zero in $CH_{132}(\epsilon)$.

□

The above lemma says that each term of $\partial_{132}^\epsilon(a)$ is equivalent to zero in CH_{132} , and so we do not get zero divisors from the *generators* that are in Q_{12}^ϵ . The following corollary shows that no zero divisors come from any combination of generators of Q_{12}^ϵ , for $\epsilon \neq CCC$.

Corollary 5.4.2. *If $a \in Q_{12}^\epsilon$, $\epsilon \neq CCC$, then $\partial_{132}^\epsilon(a)$ is not a zero divisor.*

We would be done if we knew that if $CH_{132}(\epsilon)$ has zero divisors if and only if there exists a generator a of \mathbb{T}_{132} such that $\partial_{132}^\epsilon(a)$ is a zero divisor. This follows from the fact that we have shown that for any generator of \mathbb{T}_{132} , the ∂ calculation of that generator has linear terms (in which case it does not give a zero divisor) or it is in Q_{12}^ϵ and so its image under ∂ is not a zero divisor by the above lemma. It is not currently known if this is true, and so we proceed to investigate the full algebra for all 27 cases.

Conjecture 5.4.2. *CH_{132} has no zero divisors for all but one of the possible minimal augmentations of the 6-copy of $(-4, 3)$.*

Basis for conjecture. Knowing that any zero divisors of CH_{132} must come from generators whose ∂ calculation has linear terms makes the calculations of the full algebra much simpler. All 27 cases have been checked by hand,

and in each case we find that for $\epsilon \neq CCC$, $CH_{132}(\epsilon)$ is freely generated by some subset of the generators of \mathbb{T}_{132} . For example, when $\epsilon = AAA$, $CH_{132}(AAA)$ is freely generated by vertices of the form $a_{even,n}$, $b_{even,n}$, $c_{even,n}$ for $n = 1, \dots, 5$, and by all vertices of the form $a_{m,6}$, $b_{m,6}$, $c_{m,6}$. Since there are no relations in this algebra, there are no zero divisors.

When $\epsilon = CCC$, in $CH_{132}(CCC)$ we have that $b_{m,1} \equiv b_{m+3,1}$ and $c_{m,1} \equiv c_{m+1,1}$ for all m . There are no other relations involving the $b_{m,1}$, $c_{m,1}$ terms. In particular, we have that $\partial_{132}^{CCC}(a_{4,2}) = b_{3,1}c_{4,1}$, and neither of $b_{3,1}$, $c_{4,1}$ are in the image of ∂_{132}^{CCC} . Thus $CH_{132}(CCC)$ has zero divisors.

In this way, it appears that CH_{132} has no zero divisors for 26 out of the 27 possible minimal augmentations for the link. \square

Remark 5.4.3. Note that if we knew that any isotopy taking L_{123} to L_{132} induces a permutation of the minimal augmentation classes, then we would be done since $z_{123} \geq 2$ and $z_{132} = 1$. (As the z invariant would then be well-defined on augmentation classes of minimal augmentations.) While this may be true, it is currently unknown. Therefore, we outline the proof below that $z_{123} > z_{132}$ for all possible augmentations of $\mathbb{I}_6(6, 14)$. \diamond

Basis for Conjecture 5.4.1. Assuming that Conjecture 5.4.2 is true, we have that in 26 of the 27 cases above, CH_{132} does not have zero divisors. If we add augmented vertices to any of those 26 cases, we have verified by hand that there still will be no zero divisors. Therefore, there can only be zero divisors by adding to the CCC case above. Recall that the augmentation C

is given by $x_{4,3} = x_{5,3} = x_{6,3} = x_{7,3} = x_{8,3} = x_{9,3} = x_{10,3} = x_{11,3} = 1$ and all other $x_{i,j} = 0$. The question is what other $x_{i,j}$ can equal 1 and still give a proper augmentation of the algebra. It can be shown directly from the ∂ calculation that $x_{2,1} = 1 \Leftrightarrow x_{4,1} = 1 \Leftrightarrow x_{6,1} = 1 \Leftrightarrow x_{odd,1} = 0$. In this case, we also get that $x_{j,2} = 1$ for all possible j values. Call this augmentation C_2 . It can also be shown that $x_{3,1} = 1 \Leftrightarrow x_{5,1} = 1 \Leftrightarrow x_{7,1} = 1 \Leftrightarrow x_{9,1} = 1 \Leftrightarrow x_{even,1} = 0$. In this case, we also get that $x_{j,2} = 1$ for all possible j values. Call this augmentation C_3 . We can further show that if any $x_{j,2} = 1$ then $x_{j,2} = 1$ for all possible j values, and so we must have either C_2 or C_3 . It is a straightforward calculation to see that any combination of C , C_2 , and C_3 has no zero divisors except for CCC as already shown above. Essentially, we have that if any augmented vertices are added to C in the “front” of the diagram (i.e. closer to the left cusps), then there are no zero divisors. If we only add augmented vertices to the “back” of the diagram (i.e. further away from the left cusps), there will be zero divisors. Call these augmentations C_4, \dots, C_n . Any combination of C, C_4, \dots, C_n will have zero divisors. Mixing all of these together, we have that if any component has the C_2 or C_3 augmentation, then there are no zero divisors, otherwise there are zero divisors. It can be shown that in CH_{123} , the combinations that do not include C_2 or C_3 also have zero divisors. But, the triple of any of the new augmentations will also have zero divisors by Lemma 5.4.3.

We now turn our attention to augmentation classes. While we do not know how many augmentation classes the above described augmentations

represent, we do know enough to show that $z_{123} \neq z_{132}$. As shown above, when vertices are added “in back” of the CCC augmentation, there are zero divisors in CH_{132} . Let x equal the number of distinct augmentation classes of this form, including CCC . Thus, $z_{132} = x$, and $x \geq 1$. It was also shown above that all x of these augmentation classes yield zero divisors in CH_{132} , but so does $[AAA]$. The x classes representing the CCC augmentation and the CCC augmentation with vertices added in back are each not equivalent to the class $[AAA]$ by Lemma 2.3.3 since the first x yield zero divisors in CH_{132} and $[AAA]$ does not. Thus $z_{132} \geq x + 1$, and so $z_{123} > z_{132}$. Therefore, $L_{123} \neq L_{132}$, and only cyclic permutations are possible for the n -copy of $(-4, 3)$. \square

While the proofs in this section are limited to showing that only cyclic permutations are possible for $(-4, 3)$, they could be easily adapted to prove this result for a general negative $(p, 3)$ -torus knot. For a fixed projection, a negative $(p, 3)$ -torus knot will not have a proper augmentation, and its double will have three minimal augmentations. The proof of Conjecture 5.4.2 will easily adapt to a general p value. Note that Lemma 5.4.3 is already proved for a general negative (p, q) -torus knot. A strategy for proving the following conjecture will then be complete by a similar argument to the one for $(-4, 3)$ above.

Conjecture 5.4.3. *Only cyclic permutations are possible for the n -copy of a negative $(p, 3)$ -torus knot.*

5.5 Other Negative (p, q) -Torus Links

We now describe a possible strategy for showing that only cyclic permutations are possible for negative (p, q) -torus links. The first step will be to show that any augmentation of the double interval algebra associated to a projection of a negative (p, q) -torus knot is either a minimal augmentation, or a minimal augmentation with added vertices.

Conjecture 5.5.1. *Let K be a negative (p, q) -torus knot. If q is even, K has a unique minimal augmentation and if q is odd, K does not have an augmentation. The double of K has q minimal augmentations if q is even, and $q - 1$ minimal augmentations if q is odd.*

When q is even, although there is a unique minimal augmentation ϵ , the algebras $CH_{123}(\epsilon)$ and $CH_{132}(\epsilon)$ both have divisors of zero and so we cannot distinguish the links. This unique augmentation will have a form similar to that of the C augmentation described in the proof of $(-4, 3)$ above, and so a “CCC” type augmentation of the 3-copy will have zero divisors by a similar argument to that for $(-4, 3)$. It will then be necessary to work with the 2-copy, and then the 3-copy of the 2-copy as for $(-4, 3)$.

Notice that the reason that the CCC augmentation for $2(-4, 3)$ gives zero divisors in CH_{132} is because when viewed on the front projection of $2(-4, 3)$, these augmentation are all aligned vertically, making it impossible for the differential of any generator to equal a single generator (i.e. no generator is equal to zero in CH_{132}). Of the q (or $q - 1$) minimal augmentations in the

conjecture above, only one of these will have the augmentations vertically aligned in the front projection. For every negative (p, q) -torus knot checked so far, this is the only minimal augmentation that yields zero divisors in CH_{132} . A proof is not yet completed, and will need to be more general than the one for $2(-4, 3)$ as the number of augmentations may be prohibitively large. Once this is proved, it will remain to show that when adding vertices to one of the minimal augmentations, a result similar to the one for $2(-4, 3)$ is found. That is, when adding “in back” of the aligned triple (the analog of CCC) there will be x equivalence classes that yield zero divisors in CH_{132} . Further, those same x equivalence classes will give zero divisors in CH_{123} , as will the triple of any augmentation of $2(-p, q)$. This last fact is known in general since the proof of Lemma 5.4.3 does not use anything that is particular to $2(-4, 3)$. The proof will then be complete as we will have that $z_{123} > z_{132}$.

The above outlines a strategy for showing noncyclic permutations are not possible for any negative (p, q) -torus link. It is clear that this *DGA* approach is lengthy and difficult. It would be interesting to know if there exists a convex surface argument that would show this result more easily.

Bibliography

- [1] Y. Chekanov, *Differential algebra of Legendrian links*, Inventiones mathematicae **150** (2002), 441–483.
- [2] J. Dalton, J. Etnyre, and L. Traynor, *Legendrian and transversal torus links*, In production.
- [3] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet’s work*, Ann. Inst. Fourier (1992), 165–192.
- [4] Y. Eliashberg and M. Fraser, *Classification of topologically trivial Legendrian knots*, Geometry, Topology, and Dynamics, CRM Proc. Lecture Notes, 15, Amer. Math. Soc., Providence, RI, 1998, pp. 17–51.
- [5] J. Epstein, *On the invariants and isotopies of Legendrian and transversal knots*, Ph.D. thesis, University of California, Davis, 1997.
- [6] J. Etnyre, *Transversal torus knots*, Geometry and Topology **3** (1999), 253–268.

- [7] ———, *Convex surfaces in contact geometry: class notes*, 2004, Available online at <http://www.math.gatech.edu/~etnyre/preprints/papers/surfaces.pdf>.
- [8] ———, *Legendrian and transversal knots*, Handbook of Knot Theory (W. Menasco and M. Thistlethwaite, eds.), Elsevier, Amsterdam, 2005, pp. 105–185.
- [9] J. Etnyre and K. Honda, *Knots and contact geometry 1: torus knots and the figure eight knot*, Journal of Symplectic Geometry **1** (2001), 63–120.
- [10] J. Etnyre, L. Ng, and J. Sabloff, *Invariants of Legendrian knots and coherent orientations*, Journal of Symplectic Geometry **1** (2002), no. 2, 321–367.
- [11] D. Fuchs, *Chekanov-Eliashberg invariant of Legendrian knots: existence of augmentations*, Journal of Geometry and Physics **47** (2003), 43–65.
- [12] D. Fuchs and S. Tabachnikov, *Invariants of Legendrian and transverse knots in the standard contact space*, Topology **36** (1997), 1025–1053.
- [13] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helvetica **66** (1991), 637–677.
- [14] ———, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Inventiones mathematicae **141** (2000), 615–689.

- [15] K. Honda, *On the classification of tight contact structures I*, Geometry & Topology **4** (2000), 309–368.
- [16] Y. Kanda, *On the Thurston-Bennequin invariant of Legendrian knots and non exactness of Bennequin’s inequality*, Inventiones mathematicae **133** (1998), 227–242.
- [17] K. Mishachev, *The n -copy of a topologically trivial Legendrian knot*, Journal of Symplectic Geometry **1** (2002), 659–682.
- [18] L. Ng, *Computable Legendrian invariants*, Topology **42** (2003), no. 1, 55–82.
- [19] L. Ng and J. Sabloff, *The correspondence between augmentations and rulings for Legendrian knots*, Pacific J. Math **224** (2006), no. 1, 141–150.